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[This question paper contains 4 printed pages.]

Your Roll No.............. L

Sr. No. of Question Paper : 1772

Unique Paper Code : 2352013603

Name of the Paper : DSC-18 Complex Analysis

Name of the Course : **Bachelor of Science (Honours Course) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
2. **All** questions carry equal marks.
3. Attempt **two** parts from each question.

1. (a) (i) Solve $e^z = 3$, where z is any complex number.

(ii) Find the values of $\text{Log}(1 + i)^2$ and $\text{Log}(-1 + i)^2$ and justify the truth of $\text{Log}(z_1.z_2) = \text{Log}(z_1) + \text{Log}(z_2)$ for $z_1, z_2 \in \mathbb{C}$.

(b) Prove that if $f'(z) = 0$ everywhere in a domain D, then f(z) must be constant throughout D.

(c) Show that each of the following functions is nowhere analytic :

(i) $f(z) = xy + iy$

(ii) $f(z) = 2xy + i(x^2 - y^2)$

2. (a) Prove that if the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some $\in$-neighbourhood of a point $z_0 = x_0 + iy_0$, and the following conditions hold true :

(i) The first order derivatives of the functions u and v with respect to x and y exist everywhere in the neighbourhood;

(ii) Those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy- Riemann equations at (x_0, y_0),

Then $f'(z)$ exists and $f'(z) = u_x + iv_x$ at $z = z_0$.

(b) Find the image of the domain $x > 0$, $y > 0$, $xy < 0$ under the mapping $w = z^2$.

(c) If z_0 and w_0 are points in the z and w planes, respectively, then prove that

(i) $\lim\limits_{z\to\infty} f(z) = w_0$ if and only if $\lim\limits_{z\to 0} f\left(\frac{1}{z}\right) = w_0$.

(ii) $\lim\limits_{z\to\infty} f(z) = \infty$ if and only if $\lim\limits_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$.

3. (a) Prove that if w(t) is piecewise continuous complex-valued function defined on an interval $a \le t \le b$, the :

$$\left|\int_a^b w(z)\,dt\right| \le \int_a^b |w(t)|\,dt.$$

Find an upper bound for $\left|\int_C \frac{z+4}{z^3-1}\,dz\right|$, where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$.

(b) Suppose that a function f(z) is continuous in a domain D. Prove that if f(z) has an antiderivative F(z) throughout D, then the value of the integral of f(z) along contours lying entire y in D and extending from any fixed point z_1 to any other fixed point z_2 is

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

Use it to find the value if $\int_0^{\pi+2t} \cos\left(\frac{z}{2}\right) dz$.

(c) State Cauchy-Goursat theorem. Use it to show that

$$\int_C f(z)\,dz = 0, \text{ when}$$

(i) $f(z) - \dfrac{1}{z^2+2z+2}$ and C is the unit circle $|z| = 1$.

(ii) $f(z) - \dfrac{5z+7}{z^2-4z+3}$ and C is the unit circle $|z-2| = 2$.

4. (a) For a function f given to be analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R, if M_R denotes the maximum value of $|f(z)|$ on C_R, then prove that $\left|f^n(z_0)\right| \le \dfrac{n!M_R}{R^n}$.

Hence, show that if f is an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number, then $f(z) = a_1 z$, where a_1 is a complex constant.

(b) Evaluate $\int_C \dfrac{3z+1}{z(z-2)^2}dz$, where C is the figure-eight contour formed by two circles with centres at (0,0) and (2,0) respectively with radii 0.5 and 1.5 respectively, described in the positive direction,

(c) State and prove Liouville's theorem. Apply this to show that if f(z) is entire and the function $u(x, y) = \mathrm{Re}[f(z)]$ is bounded above by u_0 at all points (x, y) in the xy plane, then u(x, y) must be constant throughout the plane.

5. (a) Give all the Laurent series expansions in the powers of z for the function $f(z) = \dfrac{1}{z^2(1-z)}$ and specify the domains in which those expansions are valid.

(b) Suppose that $z_n = x_n + iy_n$ (n = 1,2,3,.....) and $S = X + iY$. Then, prove that $\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. Also show that if $\sum_{n=1}^{\infty} z_n = S$ then $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}$.

(c) State Taylor's theorem and derive Taylor's series representations for the following :

(i) $\dfrac{z}{\left(z^4+9\right)} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^{2n+2}} z^{4n+1}$ where, $|z| < \sqrt{3}$

(ii) $\dfrac{\sin\left(z^2\right)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} \ldots\ldots\ldots$, when $z \neq 0$.

6. (a) State Cauchy's residue theorem and use it to evaluate the integral

$$\int \frac{dz}{z^3(z+4)}$$

over the circle $|z+2| = 3$ oriented anticlockwise.

(b) Define the terms: isolated and non-isolated singular points. Write the principal part of each of the following functions at their respective isolated singular points and determine whether that point is a pole, a removable singular point, or an essential singular point.

(i) $z \exp\left(\dfrac{1}{z}\right)$

(ii) $\dfrac{1-\cos z}{z^2}$

(iii) $\dfrac{1}{(2-z)^3}$.

(c) Use residues to evaluate $\int_0^{2\pi} \dfrac{d\theta}{5+4\sin\theta}$.