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[This question paper contains 2 printed pages.]

Sr. No. of Question Paper : 2448

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Unique Paper Code : 2352572401

Name of the Paper : Abstract Algebra

Name of the Course : **B.A./B.Sc. (Programme) with Mathematics as Non-Major/ Minor – DSC**

Semester : IV

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. The use of a calculator is not allowed.

1. (a) Define an abelian group. Show that the set of all 2×2 matrices with real entries is (7.5)
not a group under multiplication.
- (b) Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \neq 0, a \in \mathbb{R} \right\}$. Prove that G is a group under matrix (7.5)
multiplication.
- (c) Find the inverse of $\begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix} \in \text{SL}(2, \mathbb{Z}_5)$. (7.5)
2. (a) What is a subgroup? Let G be an abelian group with identity e . Then show that (7.5)
 $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G .
- (b) Let G be an abelian group and H and K be subgroups of G . Then show that $HK =$ (7.5)
 $\{hk \mid h \in H, k \in K\}$ is a subgroup of G .
- (c) In the group $\mathbb{Z}_{12}$, find $|a|$, $|b|$ and $|a + b|$ when $a = 5$, $b = 4$. (7.5)
3. (a) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 6 & 2 & 5 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{bmatrix}$. (7.5)
Compute α^{-1} , β^{-1} , $O(\alpha)$, $O(\beta)$ and $\beta\alpha$.
- (b) List the elements of the subgroups $\langle 3 \rangle$ and $\langle 15 \rangle$ in $\mathbb{Z}_{18}$. Let a be a group element (7.5)
of order 18. List the elements of the subgroups $\langle a^3 \rangle$, and $\langle a^{15} \rangle$.
- (c) State Lagrange's Theorem. Suppose that K is a proper subgroup of H and H is a (7.5)
proper subgroup of G . If $|K| = 42$ and $|G| = 420$, what are the possible orders
of H .

P.T.O.

4. (a) Prove that the group $SL(2, \mathbf{R})$ of 2×2 matrices with determinant 1 is a normal subgroup of $GL(2, \mathbf{R})$, the group of 2×2 matrices with nonzero determinant. (7.5)
- (b) Define factor group. What is the order of the factor group $Z_{60}/\langle 15 \rangle$. (7.5)
- (c) Let ϕ be a homomorphism from a group G to a group $\bar{G}$ and let H be a subgroup of G . Then show that if H is cyclic then $\phi(H)$ is also cyclic. Check whether the mapping $\phi(x) = x^2$ from $\mathbf{R}$, the real numbers under addition, to itself is a homomorphism? (7.5)
5. (a) (i) Prove that the set $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$ is a subring of the ring of all 2×2 matrices over $\mathbb{Z}$. (7.5)
- (ii) Prove that a unit of a ring divides every element of the ring.
- (b) Prove that the ring $\mathbb{Z}_p$ of integers modulo p where p is a prime is an integral domain. Is $\mathbb{Z} \oplus \mathbb{Z}$ an Integral Domain? Justify. (7.5)
- (c) Prove that characteristic of an Integral Domain is 0 or a prime number. (7.5)
6. (a) Let $f: R \rightarrow S$ be a ring homomorphism. Prove that (7.5)
- (i) $f(A)$ is a subring of S where A is a subring of R .
- (ii) if A is an ideal and f is onto S , then $f(A)$ is an ideal of S .
- (b) (i) Prove that every ring homomorphism f from Z_n to itself has the form $f(x) = ax$, where $a^2 = a$. (7.5)
- (ii) Prove or disprove that the correspondence $x \rightarrow 5x$ from Z_5 to Z_{10} is a ring homomorphism.
- (c) Let $R[x]$ denote the set of all polynomials with real coefficients and let A denote the subset of all polynomials with constant term 0. Prove that A is an ideal of $R[x]$ and $A = \langle x \rangle$. (7.5)