(b) Let V be an inner product space, and let S = $\{v_1, v_2, v_3, \ldots\ldots\ldots v_n\}$ be an orthonormal subset of V. Prove the Bessel's Inequality :

$$||x||^2 \geq \sum_{i=1}^{n} \left|\langle x, v_i\rangle\right|^2 \quad \text{for any } x \in V.$$

(c) Prove that an orthogonal subset of a finite dimensional inner product space V can extended to a orthonormal basis for V. Hence, or otherwise prove that for any subspace W of a finite - dimensional inner product space V, dim(V) = dim(W) + dim($W^\perp$). (3+3,6,6)

6. (a) Let T be a linear operator on a complex finite dimensional inner product space V with an adjoint T*. Prove that

(i) T is self-adjoint if and only if $<Tx, x>$ is real for all $x \in V$.

(ii) If $<Tx, x> = 0$ for all $x \in V$, then $T = T_0$, the zero operator on V.

(b) Let $V = M_{2\times2}(\mathbb{R})$ and $T: V \to V$ be a linear operator given by $T(A) = A^T$. Determine whether T is normal, self-adjoint, or neither. If possible, produce an orthogonal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(c) Let U be a Unitary operator on an inner product space V and let W be a finite dimensional U-invariant subspace of V. Then, prove that

(i) U(W) = W

(ii) $W^\perp$ is U- invariant. (3+3.5,6.5,3+3.5)

[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1345 **J**

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra-II

Name of the Course : **B.Sc. (H) Mathematics (CBCS-LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the questions are compulsory.
3. Attempt any **two** parts from each question.
4. Marks of each part are indicated.

1. (a) Let F be a field. Show F[x] is a principal ideal domain.

(b) (i) Is $x^3 + 1$ irreducible over $\mathbb{Z}_7$.

(ii) Show $x^4 + 1$ is reducible over $\mathbb{R}$.

(c) State and Prove Gauss Lemma. (6,3×2,6)

2. (a) Show $\frac{\mathbb{R}[x]}{\langle x^2+1\rangle} \cong \mathbb{C}$. Hence, Prove $\langle x^2+1\rangle$ is maximal in $\mathbb{R}[x]$.

(b) Is every Unique Factorization Domain a Principal Ideal Domain? Prove or give a Counterexample.

(c) Show $\mathbb{Z}\left[\sqrt{-5}\right]$ is not a Unique Factorization Domain. (Consider the factorization of the element 6). (6.5+6.5+6.5)

3. (a) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows

$f_1(x, y, z) = x - 2y$, $f_2(x, y, z) = x + y + z$ and $f_3(x, y, z) = y - 3z$.

Prove that $\{f_1, f_2, f_3\}$ is a basis of V^* and then find a basis for V for which it is the dual basis.

(b) Show that the matrix $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ is diagonalizable and hence find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(c) Test the linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(a, b, c) = (4a + c, 2a + 3b + 2c, a + 4c)$ for diagonalizability and if it is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix. (6,6,6)

4. (a) For the linear operator $T: \mathbb{R}^4 \to \mathbb{R}^4$, $T(a, b, c, d) = (a + b, b - c, a + c, a + d)$, determine the T – cyclic subspace W of $\mathbb{R}^4$ generated by $e_1 = (1,0,0,0)$. Also, find an ordered basis of W and characteristic polynomial of the operator T_W.

(b) State Cayley-Hamilton theory and verify it for the linear operator

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(a, b, c) = (a + b, b + c, b - c)$.

(c) Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard bases β and γ. Define $T: V \to W$ by $T(p(x)) = (p(0) - 2p(1), p(0) + p'(0))$, where $p'(x)$ is the derivative of $p(x)$.

(i) For $f(a, b) = a - 2b$, $f \in W^*$, compute $T^t(f)$.

(ii) Compute $\left[T^t\right]_{\gamma^*}^{\beta^*}$ by finding scalars a, b, c and d such that $T^t(f_1) = af_3 + bf_4$ and $T^t(f_2) = cf_3 + df_4$, where $\beta^* = \{f_3, f_4\}$ and $\gamma^* = \{f_1, f_2\}$ are the respective dual bases. (6.5,6.5,3+3.5)

5. (a) Show that in a complex inner product space V over a field F. If $x, y \in V$, then following identities hold :

(i) $\langle x, y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$ if $F = \mathbb{R}$

(ii) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} \|x+i^k y\|^2$ if $F = \mathbb{C}$, where $i^2 = -1$.