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Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **8082**

Unique Paper Code : **2353010013**

Name of the Paper : **Mathematical Finance**

Type of the Paper : **DSE**

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

***All* questions are compulsory and carry equal marks.**

Use of scientific calculator, basic calculator and normal distribution tables all are allowed.

1. Attempt any *two* parts :

(*a*) Define the term derivative contract with an example. What is the difference between a long position and a short position in a forward contract ? Suppose that on May 01, 2025, the treasurer of a U.S. Corporation knows that the Corporation will pay £ 10 million in 6 months (i.e., on November 01, 2025). The bid and ask 6-month forward quotes or the exchange rate between USD and GBP (USD per GBP) on May 01, 2025, are 1.3091 and 1.3099, respectively. How can the corporation hedge its position against exchange rate moves ?

(*b*) What are the *three* types of traders ? Explain the arbitrage opportunity when the price of a dually listed technology company stock is \$40 (USD) on the NASDAQ and ₹ 3,500 (INR) on the National Stock Exchange of India (NSE). Assume that the exchange rate is ₹ 83 per U.S. dollar. Explain what is likely to happen to prices as traders take advantage of this opportunity.

(*c*) Let R_c denote rates of interest with continuous compounding and R_m denote equivalent rates with compounding m times per annum. Derive the relation between R_2 and R_4. An Indian investor is comparing two fixed-income investment options. One investment offers a semi-annual interest rate of 6% per annum, while another offers a quarterly interest rate. What is the equivalent quarterly interest rate that would yield the same return as the 6% semi-annual rate ? Express your answer as a percentage rounded to two decimal places.

2. Attempt any *two* parts :

(*a*) Define bond price, bond yield, and par yield. Suppose that the 6-month. 12-month, 18-month. and 24-month zero rates are 4%, 5.5%, 6.2% and

6.8% per annum, respectively (with continuous compounding). What is the bond price and 2-year par yield ? (You can use the exponential values : $e^{-0.02} = 0.9802$. $e^{-0.055} = 0.9465$, $e^{-0.0930} = 0.9112$, $e^{-0.136} = 0.8728$)

(*b*) If R_1 and R_2 are the zero rates for maturities T_1 and T_2, respectively and R_F is the forward interest rate for the period of time between T_1 and T_2. Write an expression for R_F. Suppose that the risk-free zero-coupon interest rates for 1 year, 2 years and 3 years are 4.0%, 4.4% and 4.8%, respectively. Calculate forward interest rates for the second and third years. Explain the following statement : "When the zero curve is upward-sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward-sloping, the reverse is true."

(*c*) Define the duration of a bond. What is the duration of a zero-coupon bond ? Consider a 5-year bond issued by an Indian company with a face value of ₹ 1,000 and an annual coupon rate of 8%, paid yearly. If the yield on the bond is 9% per annum with continuous compounding, what is the duration of the bond in years ? (You can use the exponential values : $e^{-0.09} = 0.9139$, $e^{-0.18} = 0.8353$, $e^{-0.27} = 0.7634$, $e^{-0.36} = 0.6977$, $e^{-0.45} = 0.6376$)

3. Attempt any *two* parts :

(*a*) Distinguish between a European option and an American option. Additionally, explain the concepts of a long position in a call option and a short position in a put option. Illustrate each with their respective payoff functions and payoff diagrams.

(*b*) Show that the lower bound for the price of a European call option on a non-dividend-paying stock when the stock price is S_0, the strike price is K, the risk-free interest rate is $r\%$, and the time to expiration of option is T, is given by $S_0 - Ke^{-rT}$. Also, what is the lower bound for the price of a one-month European put option on a non-dividend paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum ? (You can use the exponential value : $e^{-0.0049} = 0.995$).

(*c*) Derive the put-call parity relation for a European stock option on a non-dividend paying stock. The price of a non-dividend paying stock is \$19 and the price of a three-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price of \$20 ? (You can use the exponential value : $e^{-0.01} = 0.990$).

4. Attempt any *two* parts :

(*a*) Explain how a bull spread can be created using two European call options on a stock ? Write the payoff table along with profit diagram corresponding to this trading strategy. A call option with a strike price of \$50 costs \$2. A put option with a strike price of \$45 costs \$3. Explain how a strangle can be created from these two options. What is the pattern of profits from the strangle ?

(*b*) Define briefly the no-arbitrage valuation approach to valuing a European option. A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-month European call option with a strike price of \$39 ? (You can use the exponential value : $e^{-0.0066} = 0.993$).

(*c*) Define briefly what is meant by risk neutral world in the context of valuing a European option using a binomial tree. A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100 ? (You can use the exponential values : $e^{0.04} = 1.040$, $e^{-0.08} = 0.923$).

5. Attempt any *two* parts :

(*a*) Stock price in the Black-Scholes model satisfies :

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where $\phi(m, v)$ denotes a normal distribution with mean m and variance v. Find $P(S_T < K)$.

(*b*) Stock price in the Black-Scholes model satisfies :

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where $\phi(m, v)$ denotes a normal distribution with mean m and variance v. Find $E[S_T]$ and $E\left[S_T^2\right]$.

(*c*) Write a short note on stop-loss strategy. Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

6. Attempt any *two* parts :

(*a*) Let V be a lognormal random variable with ω being the standard deviation of ln V. Prove that $E[\max(K - V, 0)] = KN(-d_2) - E(V)N(-d_1)$, where

$$d_1 = \frac{\ln\left[\frac{E(V)}{K}\right] + \frac{\omega^2}{2}}{\omega},$$

$$d_2 = \frac{\ln\left[\frac{E(V)}{K}\right] - \frac{\omega^2}{2}}{\omega},$$

and E denotes the expected value. Use this result to derive the Black-Scholes formula for the price of a European put option on a non-dividend paying stock.

(*b*) Define delta and gamma of a portfolio. Suppose a financial institution has the following three positions on a stock :

(*i*) A long position in 100,000 call options with strike price $55 and an expiration date in 3 months. The delta of each option is 0.533.

(*ii*) A short position in 200,000 call options with strike price $56 and an expiration date in 5 months. The delta of each option is 0.468.

(*iii*) A short position in 50,000 put options with strike price $56 and an expiration date in 2 months. The delta of each option is –0.508.

Calculate delta of the whole portfolio and how portfolio can be made delta-neutral. Suppose that gamma of this delta-neutral portfolio of options on an asset is –10,000. If there is a change of +3 or –3 in the price of an asset over a short period of time, what is the change in the value of the portfolio ?

(c) In Black-Scholes model, the stock prices follow the Geometric Brownian motion given by :

$$dS = \mu S\,dt + \sigma S\,dz$$

Using Itô's Lemma on ln S, show that :

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$