[This question paper contains 7 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **8083** I

Unique Paper Code : 2353010014

Name of the Paper : Integral Transforms

Name of the Course : **B.Sc.(H) Mathematics (DSE)**

Semester : VI

Time : 3 Hours ***Maximum Marks : 90***

Instructions for Candidates :

(i) Write your Roll No. on the top immediately on receipt of this question paper.

(ii) Attempt **all** questions by selecting **two** Parts from each question.

(iii) **All** questions carry equal marks.

(iv) **All** Parts from each question carry **7.5** marks.

(v) Parts of questions to be attempet together.

(vi) Use of calculator is not allowed.

P.T.O.

1. (a) Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.... = \frac{\pi^2}{8}$.

(b) Obtain the complex Fourier series expansion for the function $f(x) = e^x$ in $[-\pi, \pi]$. Use the Parseval relation to show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(1+k^2)} = \pi \coth \pi.$$

(c) Using the Fourier integral theorem, show that if

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x < 0, x > \pi \end{cases}$$

then

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos ax + \cos(a(\pi - x))}{1-a^2} da, -\infty < x < \infty.$$

Hence show that $\int_0^{\infty} \frac{\cos\left(\frac{a\pi}{2}\right)}{1-a^2} da = \frac{\pi}{2}$.

2. (a) Show that the Fourier series of the triangular function defined by

$$f(x)=\begin{cases}\dfrac{2x}{\pi}, & 0\le x\le \pi/2\\ \dfrac{2}{\pi}(\pi-x), & \pi/2\le x\le \pi\end{cases}$$

is

$$f(x)=\frac{8}{\pi^2}\left\{\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}+\ldots\right\}.$$

Further show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

(b) State and prove Riemann-Lebesgue Lemma.

(c) Determine the Fourier series expansion of x^2 for $-\pi<x<\pi$, by performing the integration of the appropriate Fourier series.

3. (a) If $\bar{f}(s)$ and $\bar{g}(s)$ are the Laplace transform of f (t) and g (t) respectively, then prove that :

(i) $\mathcal{L}[H(t-a)f(t-a)]=e^{-as}\mathcal{L}\{f(t)\}$

(ii) $\mathcal{L}[H(t-a)g(t)]=e^{-as}\mathcal{L}\{g(t+a)\}$.

Also, find out the Laplace transform of the function

$$f(t)=\begin{cases}0, & t<2\\ t-2, & t\ge 2\end{cases}.$$

(b) Find the inverse Laplace transform of the following functions :

(i) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

(ii) $\dfrac{1}{s(s+a)}$

(c) Define convolution of two functions in Laplace transform and then prove the following :

(i) $f*g=g*f$

(ii) $\frac{d}{dt}[(f*g)(t)]=(f'*g)(t)+f(0)g(t)$

4. (a) Define the Fourier transform of a function f(t) and show that :

$$f\{\exp(-a|x|)\}=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+k^2}\right), a>0$$

(b) Define the Fourier cosine transform of a function f(x). Let f (x) and its first derivative vanish as $x \to \infty$. If $F_c(k)$ is the Fourier transform, then prove that:

$$f_C\left[f''(x)\right] = -k^2 f_C(k) - \sqrt{\frac{2}{\pi}} f'(0)$$

(c) Prove the following :

(i) $\int_{-\infty}^{\infty} (f * g)(x)\,dx = \int_{-\infty}^{\infty} f(u)\,du \int_{-\infty}^{\infty} g(v)\,dv$

(ii) $\int_{-\infty}^{\infty} F(k) g(k)\,dk = \int_{-\infty}^{\infty} f(x) G(x)\,dx$,
where F(k) and G(x) are Fourier transforms of f(x) and g(k) respectively.

5. (a) Find the solution of the ordinary differential equation

$$-\frac{d^2u}{dx^2} + a^2 u = f(x), \quad -\infty < x < \infty$$

by the Fourier transform method.

(b) Let a string of length π be fixed at both the ends. The motion of the string due to a force acting on it is governed by

$$u_{tt} = c^2 u_{xx} + f(x,t), 0 < x < \pi, \quad t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < \pi,$$
$$u(0,t) = 0 \quad u(\pi,t) = 0, \quad t > 0$$

Obtain the solution to the above equation of motion by applying the finite Fourier sine transform.

(c) Obtain the solution of the following Cauchy Problem for the Diffusion Equation using Fourier transform.

$$u_t = k u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

where k is a diffusivity constant with the initial condition

$$u(x,0) = f(x), \quad -\infty < x < \infty$$

6. (a) Use Laplace transform to find the solution of the following equation

$$x u_t + u_x = x, \quad x > 0, \quad t > 0$$

with the initial and boundary conditions

$$u(x,0) = 0 \quad \text{for} \quad x > 0,$$
$$u(0,t) = 0 \quad \text{for} \quad t > 0$$

(b) Use Laplace transform to find the solution of the system of equations

$$\frac{dx_1}{dt} - x_2 = 0,$$

$$\frac{dx_2}{dx} + 2x_1 - 3x_2 = 0$$

with $x_1(0) = 0$ and $x_2(0) = 1$.

(c) Use Laplace transform to solve the equation

$$u_t = ku_{xx}, \quad x > 0,\ t > 0$$

with the initial and boundary conditions

$$u(x,0) = 0, \qquad x > 0$$
$$u(0,t) = f(t), \qquad t > 0$$
$$u(x,t) \to 0 \quad \text{as } x \to \infty, \quad t > 0$$

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