This question paper contains 4 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **5574**

Unique Paper Code : **2352013603**

Name of the Paper : **Complex Analysis**

Name of the Course : **B.Sc. (Hons.) Mathematics (NEP-UGCF 2022)**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory and carry equal marks.

Each question contains three parts (*a*), (*b*) and (*c*), attempt only *two* parts.

1. (*a*) (*i*) Sketch the set $S = \left\{z \in \mathbf{C} : |z| \leq 1, -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}\right\}$. 3

(*ii*) Under the map $f(z) = z^2$, draw the image of the set : 4.5

$$S = \left\{z \in \mathbf{C} : |z| = 1, -\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}\right\}.$$

(*b*) Under the map $f(z) = e^z$ draw the image of the rectangular region

$$R = \left\{z = x + iy \in \mathbf{C} : 1 \leq x \leq 2, -\frac{\pi}{4} \leq y \leq \frac{\pi}{2}\right\}.$$ 7.5

(c) Does $\lim_{z \to 0} \left(\frac{z}{\bar{z}} \right)^2$ exist ? Justify your answer. 7.5

2. (a) Show that the function $f(z) = \bar{z}$ is nowhere differentiable on **C**. 7.5

(b) (i) Giving proper justification, write two differences between the functions : 3.5

$$f(x) = e^x,\ x \in \mathbf{R} \text{ and } f(z) = e^z,\ z \in \mathbf{C}$$

(ii) Solve the equation $e^z = -2$. 4

(c) Find out the values of $\sin^{-1} 2$. 7.5

3. (a) (i) Let a function f be integrable along a contour C. Prove that : 3.5

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

(ii) Define a simple arc. Give an example to explain that a same set of points may represent different arcs. 4

(b) Let C be any simple closed contour, described in the positive sense in the z plane, then compute the value of the integral :

$$\int_C \frac{s^3 + 2s}{(s - z_0)^3} ds,$$

(i) when z_0 is inside C, and

(ii) when z_0 is outside C. 7.5

(c) (i) Let $w(t)$ be a piecewise continuous complex valued function defined on an interval $a \le t \le b$, then show that : 3.5

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt.$$

(ii) Let C be the arc of the circle $|z| = 2$ such that $0 \le arg z \le \frac{\pi}{2}$.

Show that : 4

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}.$$

4. (a) State and prove Cauchy integral formula. 7.5

(b) Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = bz$, where b is a complex constant. 7.5

(c) (i) Evaluate the following integral along the line segment, say L, from 0 to $\pi + 2i$: 3.5

$$\int_L \cos(z/2)dz.$$

(ii) Evaluate the following integral :

$$\int_C \frac{z\,dz}{(9-z^2)(z+i)},$$

where C is the positively oriented circle $|z| = 2$. 4

5. (a) State Laurent's series theorem. Give two series expansions in powers of z for the function $f(z) = \frac{z+1}{z-1}$ specifying the regions in which those expansions are valid. 7.5

(b) If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $z = z_1$ $(z_1 \ne z_0)$, then prove that it is absolutely convergent at each point z in the open disk $|z-z_0| < R_1$ where $R_1 = |z_1 - z_0|$. 7.5

(c) Define residue. Evaluate the residue of the function $f(z) = \frac{e^z - 1}{z^4}$ and use it to find the integral $\int_C \frac{e^z - 1}{z^4} dz$, where C is the positively oriented unit circle $|z| = 1$. 7.5

6. (a) Define an isolated singular point. For the below given functions, find the isolated singular point and determine whether it is a removable singular point, an essential singular point, or a pole : 7.5

(i) $\frac{\sin z}{z}$ and

(ii) $\frac{e^{2z}}{(z-1)^2}$.

(b) Let z_0 be an isolated singular point of a function f. Then show that the following two statements are equivalent :

(i) z_0 is a pole of order $m(m = 1, 2, ...)$ of f;

(ii) $f(z)$ can be written in the form $f(z) = \frac{\varphi(z)}{(z - z_0)^m}$, where $\varphi(z)$ is analytic and non-zero at z_0.

Further, establish that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} \text{ when } m = 1, 2, 3, ...$$

where $\varphi^{(0)} \equiv \varphi$. 7.5

(c) Using residues, evaluate the following integral : 7.5

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} d\theta.$$