This question paper contains 4 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **5783**

Unique Paper Code : **2352011201**

Name of the Paper : **Linear Algebra**

Name of the Course : **B.Sc. (H) Mathematics**

Semester : **II / DSC**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *all* questions by selecting any *two* parts from each question.

All questions carry equal marks.

Use of calculator is not allowed.

1. (*a*) Prove that if x, y and z are mutually orthogonal vectors in $\mathbf{R}^n$, then

$$\|x + y + z\|^2 = \|x\|^2 + \|y\|^2 + \|z\|^2.$$

Also, verify the same for the vectors in $\mathbf{R}^4$, where $x = (1, 0, -1, 0)$, $y = \left(1, \sqrt{2}, 1, 1\right)$ and $z = \left(1, -\sqrt{2}, 1, 0\right)$.

(*b*) Solve the following system of linear equations using Gaussian Elimination Method :

$$3x_1 - 2x_2 + 4x_3 = -54$$

$$-x_1 + x_2 - 2x_3 = 20$$

$$5x_1 - 4x_2 + 8x_3 = -83.$$

(*c*) Using the Gauss-Jordon method, find the minimal integer values for the variables that will balance the following chemical equation :

$$a(HNO_3) + b(HCl) + c(Au) \rightarrow d(NOCl) + e(HAuCl_4) + f(H_2O).$$

P.T.O.

2. (*a*) Define a linear combination of the vectors in $\mathbf{R}^n$. Express the vectors [1, 11, –4, 11] as a linear combination of the vectors $x = [2, -4, 1, -3]$, $y = [7, -1, -1, 2]$ and $z = [3, 7, -3, 8]$.

(*b*) Consider the matrix :

$$A = \begin{bmatrix} 5 & -8 & -12 \\ -2 & 3 & 4 \\ 4 & -6 & -9 \end{bmatrix}$$

(*i*) State the Cayley-Hamilton Theorem.

(*ii*) Find the eigenvalues and eigenvectors of A.

(*iii*) Check whether the matrix A is diagonalizable or not ?

(*c*) Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ -2 & 1 & 5 \\ 3 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & -5 \\ 2 & 3 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

Compute R(AB) and (R(A))B to verify that they are equal, i.e., R(AB) = (R(A))B, for the operations :

(*i*) R : $\langle 3\rangle \leftarrow -3\langle 2\rangle + \langle 3\rangle$,

(*ii*) R : $\langle 2\rangle \leftrightarrow \langle 3\rangle$.

3. (*a*) Let $\mathbf{R}^2 = \{(a_1, a_2) : a_1, a_2 \in \mathrm{R}\}$. Prove that $\mathbf{R}^2$ is a vector space over $\mathbf{R}$ with addition and scalar multiplication defined as;

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

$$a(a_1, a_2) = (\alpha a_1, \alpha a_2) \text{ where } (a_1, a_2), (b_1, b_2) \in \mathbf{R}^2, \alpha \in \mathbf{R}.$$

(*b*) Define basis of a vector space. Prove that the following subset S of $\mathbf{R}^3$ is a spanning set of $\mathbf{R}^3$, where S = {(2, 2, 3), (–1, –2, 1), (0, 1, 0)}.

(*c*) Do the polynomials $x^3 - 2x^2 + 1$, $4x^3 - x + 3$ and $3x - 2$ generates $P_3(\mathbf{R})$?

4. (*a*) If $\{v_1, v_2, ..., v_n\}$ is a basis of a vector space V(F), then prove that every element of V can be uniquely expressed as a linear combination of $v_1, v_2, ..., v_n$.

(*b*) Define linearly dependent and independent set in $\mathbf{R}^3$. Show that set of vectors {(2, –2, 3), (0, –4, 1), (3, 1, –4)} in $\mathbf{R}^3$ is linearly dependent.

(*c*) Let W be a subspace of a finite-dimensional vector space V. Prove that W is finite-dimensional and dim(W) ≤ dim(V). Moreover, if dim(W) = dim(V), then V = W

5. (*a*) Let T : $\mathbf{R}^2 \to \mathbf{R}^2$ is a function. For each of the following parts, state why T is not a linear transformation.

(*i*) $T(a, b) = (1, b)$

(*ii*) $T(a, b) = (a, b^2)$

(*iii*) $T(a, b) = (\sin a, 0)$

(*iv*) $T(a, b) = (|a|, b)$

(*v*) $T(a, b) = (1 + a, b)$.

(*b*) Let V, W and Z be finite-dimensional vector spaces with ordered basis α, β and γ, respectively. Let T : V → W and U : W → Z be linear transformations. Then $[UT]_\alpha^\beta = [U]_\alpha^\beta \, [T]_\alpha^\beta$.

(*c*) Prove that two same dimensional finite vector spaces over the same field are isomorphic.

6. (*a*) Let V and W be vector spaces over same field F, and suppose that $\{v_1, v_2, ..., v_n\}$ is a basis for V. For $w_1, w_2, ..., w_n$ in W, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, ..., n$.

(*b*) Let $L : \mathbf{M}_{2 \times 2}(\mathbf{R}) \to \mathbf{R}^3 (\mathbf{R})$ be a linear transformation given by

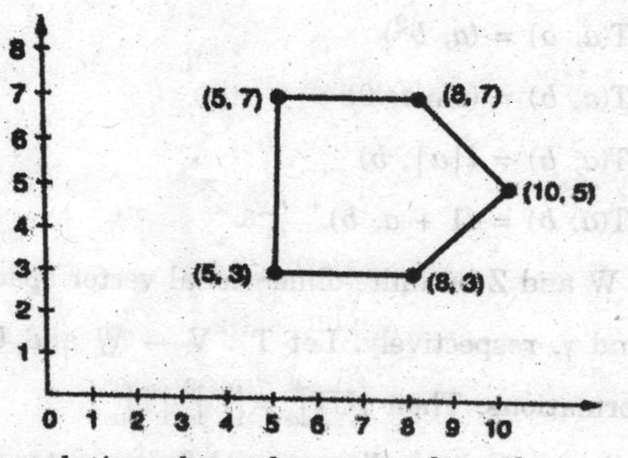
$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [a - c + 2d, 2a + b - d, - 2c + d] \text{ with}$$

$$B = \left(\begin{bmatrix} 2 & 5 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}\right);$$

$C = ([7, 0, -3], [2, -1, -2], [2, 0, 1])$.

Check whether B and C are bases for $\mathbf{M}_{2 \times 2}(\mathbf{R})$ and $\mathbf{R}^3 (\mathbf{R})$, respectively, if yes, then find matrix A for L with respect to the given bases B and C.

(*c*) For the graphic in given figure, use ordinary coordinates to find the new vertices after performing each indicated operations :



(*i*) Translation along the vector [4, –2]

(*ii*) Rotation about the origin through $\theta = 30°$

(*iii*) Reflection about the line $y = 3x$.