(b) Let G be a group of order 12. Prove that either G has a normal Sylow 3-subgroup or G is isomorphic to alternating group A_4.

(c) State Sylow's first theorem. Let G be a group of order 77. Prove that G has a normal Sylow 7-subgroup and a normal Sylow 11-subgroup. Further, prove that G is cyclic.

5. (a) State and prove the Index theorem. Use this theorem to show that there is no simple group of order 216.

(b) Let G be a finite group, and let p be the smallest prime dividing the order of G. Prove that if G has a subgroup of index p, then it must be a normal subgroup of G. Is it always true that G has a subgroup of index p? Justify.

(c) Prove that no simple group has order pqr, where p, q and r are distinct primes.

6. (a) Prove that a group G is solvable if and only if $G^{(n)} = \{e\}$ for some positive integer n, where $G^{(n)}$ is the n^{th} derived subgroup of G.

(b) Find a composition series for the symmetric group S_4.

(c) Prove that any finite p-group is nilpotent.

[This question paper contains 4 printed pages.]

Your Roll No...........

Sr. No. of Question Paper : 5514 **J**

Unique Paper Code : 2352013601

Name of the Paper : Advanced Group Theory

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI – DSC-16

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory and are of **15** marks each.
3. Attempt any **five** parts from Question **1**. Each part is of **3** marks.
4. Attempt any **two** parts from each of the Questions **2** to **6**. Each part is of **7.5** marks.

1. (i) Prove that the left regular action of a group G on itself is a faithful action.

(ii) Prove that the kernel of an action of a group G on a set A is a normal subgroup of G.

(iii) Let σ be an m-cycle in the symmetric group S_n. Find $|C_{S_n}(\sigma)|$, the order of centralizer of σ.

(iv) Is a group of order 210 simple? Justify.

(v) Define commutator subgroup G′ of a group G. Prove that G′ is normal in G.

(vi) Prove that the center of a group is a characteristic subgroup.

(vii) Define a solvable group. Prove that every abelian group is solvable.

2. (a) For a group G, consider the mapping $G \times G \to G$ given by $g.a = gag^{-1}$. Prove that this defines a group action of G on itself. Also, find the kernel of this action and the stabilizer G_x of an element $x \in G$.

(b) Let $G = D_8$ be the dihedral group of order 8 and let $A = \{1, r, r^2, r^3\}$, where r denotes a clockwise rotation of the square by $\frac{\pi}{2}$ radians. Show that the normalizer $N_{D_8}(A) = D_8$ and the centralizer $C_{D_8}(A) = A$.

(c) Let p be a prime number and let G be a group of order p^α for some $\alpha \geq 1$. Prove that G has a nontrivial center Z(G). Deduce that every group of order p^2 is abelian.

3. (a) Let the symmetric group S_n act on the set $A = \{1, \ldots\ldots, n\}$ by $\alpha.a = \alpha(a)$, $\forall\, \alpha \in S_n$ $a \in A$. Prove that this action is transitive.

(b) Let the dihedral group D_8 act via its natural action on the set $A = \{1,2,3,4\}$ consisting of four vertices of a square. Label these vertices 1,2,3,4 in a clockwise direction. Let r be the rotation of the square clockwise by $\frac{\pi}{2}$ radians and s be the reflection in the line which passes through vertices 1 and 3. Find the stabilizer of all the four vertices of square.

(c) Let G be a finite group and let $g_1, g_2, \ldots\ldots g_r$ be representatives of the distinct conjugacy classes of G not contained in the centre Z(G) of G. Then prove the class equation:

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|,$$ where $C_G(g_i)$ is the centralizer of the element g_i in G. Verify the class equation for the symmetric group S_3.

4. (a) Let G be a finite group and p be a prime dividing the order of G. Let P be a Sylow p-subgroup of G, and let n_p denote the number of Sylow p-subgroups of G. Assume that G acts transitively on the set of its Sylow p-subgroups by conjugation.

(i) Describe the orbit of P under this action.

(ii) Show that $n_p = |G : N_G(P)|$, where $N_G(P)$ denotes the normalizer of P in G.