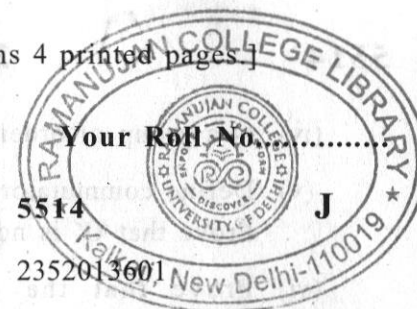


- (b) Let G be a group of order 12. Prove that either G has a normal Sylow 3-subgroup or G is isomorphic to alternating group A_4 .
- (c) State Sylow's first theorem. Let G be a group of order 77. Prove that G has a normal Sylow 7-subgroup and a normal Sylow 11-subgroup. Further, prove that G is cyclic.
5. (a) State and prove the Index theorem. Use this theorem to show that there is no simple group of order 216.
- (b) Let G be a finite group, and let p be the smallest prime dividing the order of G . Prove that if G has a subgroup of index p , then it must be a normal subgroup of G . Is it always true that G has a subgroup of index p ? Justify.
- (c) Prove that no simple group has order pqr , where p, q and r are distinct primes.
6. (a) Prove that a group G is solvable if and only if $G^{(n)} = \{e\}$ for some positive integer n , where $G^{(n)}$ is the n^{th} derived subgroup of G .
- (b) Find a composition series for the symmetric group S_4 .
- (c) Prove that any finite p -group is nilpotent.

[This question paper contains 4 printed pages.]



Sr. No. of Question Paper : 5514

Unique Paper Code : 2352013601

Name of the Paper : Advanced Group Theory

Name of the Course : B.Sc. (H) Mathematics

Semester : VI – DSC-16

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

- Write your Roll No. on the top immediately on receipt of this question paper.
- All questions are compulsory and are of 15 marks each.
- Attempt any five parts from Question 1. Each part is of 3 marks.
- Attempt any two parts from each of the Questions 2 to 6. Each part is of 7.5 marks.
- Prove that the left regular action of a group G on itself is a faithful action.
 - Prove that the kernel of an action of a group G on a set A is a normal subgroup of G .
 - Let σ be an m -cycle in the symmetric group S_n . Find $|C_{S_n}(\sigma)|$, the order of centralizer of σ .

- (iv) Is a group of order 210 simple? Justify.
- (v) Define commutator subgroup G' of a group G . Prove that G' is normal in G .
- (vi) Prove that the center of a group is a characteristic subgroup.
- (vii) Define a solvable group. Prove that every abelian group is solvable.

2. (a) For a group G , consider the mapping $G \times G \rightarrow G$ given by $g.a = gag^{-1}$. Prove that this defines a group action of G on itself. Also, find the kernel of this action and the stabilizer G_x of an element $x \in G$.

(b) Let $G = D_8$ be the dihedral group of order 8 and let $A = \{1, r, r^2, r^3\}$, where r denotes a clockwise rotation of the square by $\frac{\pi}{2}$ radians. Show that the normalizer $N_{D_8}(A) = D_8$ and the centralizer $C_{D_8}(A) = A$.

(c) Let p be a prime number and let G be a group of order p^α for some $\alpha \geq 1$. Prove that G has a nontrivial center $Z(G)$. Deduce that every group of order p^2 is abelian.

3. (a) Let the symmetric group S_n act on the set $A = \{1, \dots, n\}$ by $\alpha.a = \alpha(a)$, $\forall \alpha \in S_n, a \in A$. Prove that this action is transitive.

(b) Let the dihedral group D_8 act via its natural action on the set $A = \{1, 2, 3, 4\}$ consisting of four vertices of a square. Label these vertices 1, 2, 3, 4 in a clockwise direction. Let r be the rotation of the square clockwise by $\frac{\pi}{2}$ radians and s be the reflection in the line which passes through vertices 1 and 3. Find the stabilizer of all the four vertices of square.

(c) Let G be a finite group and let g_1, g_2, \dots, g_r be representatives of the distinct conjugacy classes of G not contained in the centre $Z(G)$ of G . Then prove the class equation:

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|, \text{ where } C_G(g_i) \text{ is the centralizer of the element } g_i \text{ in } G. \text{ Verify the class equation for the symmetric group } S_3.$$

4. (a) Let G be a finite group and p be a prime dividing the order of G . Let P be a Sylow p -subgroup of G , and let n_p denote the number of Sylow p -subgroups of G . Assume that G acts transitively on the set of its Sylow p -subgroups by conjugation.

(i) Describe the orbit of P under this action.

(ii) Show that $n_p = |G : N_G(P)|$, where $N_G(P)$ denotes the normalizer of P in G .