(b) Test for the convergence of the series whose nth term is

$$\frac{\sqrt{n+1}-\sqrt{n-1}}{n}.$$

(c) Show that the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\log(n+1)}$$

is conditionally convergent.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4765 **J**

Unique Paper Code : 2354002003

Name of the Paper : Elements of Real Analysis

Name of the Course : **Common Program Group: GE**

Semester : IV

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.

2. Attempt all questions by selecting **two** parts from each question.

3. Part of the questions to be attempted together.

4. **All** questions carry equal marks.

1. (a) Define infimum and supremum of a set. Find the infimum and supremum, if exist, of the following sets:

   (i) $\left\{2+\frac{1}{n} : n \in \mathbb{N}\right\}$.

   (ii) $\left\{\frac{x+1}{x} : x > 2\right\}$.

   (b) Let F be an Archimedean ordered field, $A \subseteq F$, and $u \in F$. Prove that $u = \sup A$ if and only if for all $\varepsilon > 0$,

   (i) for all $x \in A$, $x < u + \varepsilon$, and

   (ii) there exists $x \in A$, such that $x > u - \varepsilon$.

   (c) Define a field. Prove that the set $\mathbb{Z}$ of integers with ordinary addition and multiplication is not a field.

is not convergent by using the Cauchy criterion for convergence of series.

(c) Determine whether the Geometric series

$$\sum_{n=0}^{\infty} \frac{3^n + 4^n}{5^n}$$

converges or diverges. Find the sum of the series if it converges.

6. (a) State the Integral Test for convergence of a series. Use it to show that

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

(c) Find the limit of the sequence $\{x_n\}$, where x_n is given by

(i) $x_n = \frac{n^2}{(n)!}$

(ii) $x_n = \frac{1+2+3+\cdots+n}{n^2}$

5. (a) Prove that if the series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Give an example of a divergent series such that $\lim_{n\to\infty} a_n = 0$. What do you conclude?

(b) Show that the Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

2. (a) In any ordered field F, prove the following properties:

(i) $-|x| \le x \le |x|$.

(ii) $\max\{x, y\} = \frac{x+y+|x-y|}{2}$ and $\min\{x, y\}$ $\frac{x+y-|x-y|}{2}$.

(b) State completeness property of an ordered field. Prove that in a complete ordered field F, $\sup(A \cup B) = \max\{\sup A, \sup B\}$, where A and B are bounded above subsets of F.

(c) (i) Let

$$x_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}.$$

for $n \in \mathbb{N}$. Show that the sequence $\{x_n\}$

converges. Find its limit.

(ii) Give an example of sequences $\{x_n\}$ and $\{y_n\}$ for which $\lim_{n\to\infty} x_n = 0$,, but $\lim_{n\to\infty} x_n y_n \neq 0$.

3. (a) Define the convergence of a sequence. Using the definition of convergence, evaluate

$$\lim_{n\to\infty} \frac{3n^2 + 4n}{n^2 + 5}.$$

Find the value of the natural number n_0 for $\in = 0.01$

(b) Let $X = \{x_n\}$ and $Y = \{y_n\}$ be sequences of real numbers that converge to x and y, respectively. Prove that the sequence $X + Y$ converges to $x + y$.

(c) Let $\{x_n\}$ be a sequence of real numbers defined

inductively by $x_1 = \sqrt{7}$ and

$$x_{n+1} = \sqrt{7 + x_n},$$

for all $n \in \mathbb{N}$.. Prove that $\{x_n\}$ converges, and find its limit.

4. (a) Prove that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

(b) Define Cauchy sequence of real numbers. Further, prove that the sequence

$$\{x_n\} = \left\{\frac{n}{n^2 + 1}\right\}$$

is a Cauchy sequence.