(c) State Index Theorem. Hence or otherwise, prove that a group of order 12 is not Simple.

19/12/24 (M)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5886 **I**

Unique Paper Code : 32351502

Name of the Paper : Group Theory-2 (Core)

Name of the Course : **B.Sc. (H) Mathematics**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.

2. All questions are compulsory.

3. Question No.1 has been divided in 10 parts and each part is of 1.5 marks.

4. Each question from 2 to 6 has 3 parts and each part is of 6 marks.

P.T.O.

5. Attempt **any two** parts from each question.

1. State True (T) or False (F). Justify your answer in brief.

(a) $Aut(\mathbb{Z}_n)$ is always Abelian group, where $\mathbb{Z}_n = \{0,1,2,\ldots, n-1\}$ is group under addition modulo n.

(b) $G/Z(G)$ is isomorphic to $Aut(G)$, where $Z(G)$ is Centre of group, where $Aut\ (G)$ is group of automorphism.

(c) Group of order 9 is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, where $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is an external direct product of group $\mathbb{Z}_m$ and $\mathbb{Z}_n$..

(b) Show that two elements are conjugate in symmetric group Sn if and only if they have the same cycle type.

(c) Find a representative for each of conjugacy class of elements of order 4 in S_{12}.

6. (a) Show that a group of order pq, where P and q are distinct primes with $p < q, p \nmid (q-1)$, is cyclic.

(b) If G is a group of order 30, show that either sylow 3-subgroup or sylow 5- subgroup is normal in G.

generators r and s. Let $A = \{1, r, r^2, r^3\}$, subgroup of D_8, then show that $C_{D_8}(A) = A$.

(c) Let G be a group acting on a nonempty set A, then show that a relation on A defined by a ~ b if and only if $a = g.b$ for some $g \in G$, is an equivalence relation. Further show that, for each $a \in A$, the number of elements in the equivalence class containing a is $|G : G_a|$, the index of the stabilizer of a.

5. (a) Find all conjugate classes of the Quaternion group Q_8. Also, Verify its Class Equation.

(d) $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ is isomorphic to $\mathbb{Z}_{10}$.

(e) Order of an element (1, 3) is 6 in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

(f) Kernel of a trivial action of a group G on a set A is {e}, where e is the identity of G.

(g) 8 = 1+1+3 +3 is the class equation of group of order 8.

(h) Let G be a finite group and p be a prime that divides order of G. Then G has an element of order p.

(i) Group of order 40 has a proper normal subgroup.

(j) Group of order 35 is always cyclic.

2. (a) Let G be a group. Prove that the mapping $f(g) = g^{-1}$ $\forall g \in G$ is an automorphism if and only if G is Abelian.

(b) Prove that for every positive integer n, $Aut(\mathbb{Z}_n)$ is isomorphic to $U(n)$.

(c) Define the commutator subgroup G' of a group G. Prove that G' is a normal subgroup of G and G/G' is Abelian.

3. (a) Find the number of cyclic subgroups of order 10 in the external direct product $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$. .

(b) Prove that if a group G is the internal direct product of a finite number of subgroups $H_1, H_2, \ldots, H_n$ then G is isomorphic to the external direct product of $H_1, H_2, \ldots, H_n$

(c) Prove that every group of order p^2, where p is prime, is isomorphic to $\mathbb{Z}_{p^2}$ *or* or $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

4. (a) Let G be an abelian group of order 120 and G has exactly three elements of order 2. Determine the isomorphism class of G.

(b) Let G be a group and A be any subset of G. Define the centralizer $C_G(A)$ of A in G. Further for dihedral group D_8 of order 8 with usual