(b) If f is a one-to-one continuous mapping of a compact metric space (X, d_X) onto a metric space (Y, d_Y), then show that f^{-1} is continuous on Y and hence, f is a homeomorphism of (X, d_X) onto (Y, d_Y). (6.5)

(c) If the real-valued function f is continuous on the closed bounded interval [a, b], then prove that f is uniformly continuous on [a, b]. (6.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5832 **I**

Unique Paper Code : 32351501

Name of the Paper : BMATH511 – Metric Spaces

Name of the Course : **B.Sc. (Hons.) Mathematics (LOCF)**

Semester : V, Core

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. In the question paper, given notations have their usual meaning unless and until stated otherwise.

1. (a) (i) Define a metric space. Show that the non-negativity metric axiom is implied by the other metric axioms of coincidence, symmetry and triangle inequality.

(ii) Let X be a non-empty set.

Define for x, y ε X

$$d(x,y)=\begin{cases}0 & \text{if } x=y\\ \sqrt{2} & \text{if } x\neq y\end{cases}$$

Prove that d a metric on X. (3+3)

(b) (i) Find the open ball in $C_R[0, 1]$ centred at $f(t)=t^2$ of radius 1. Give its geometrical interpretation.

(ii) Show that (3,5) is an open ball in R. Find its centre and radius. (4+2)

(b) Let (X, d_x) be a connected metric space and f: $(X, d_X) \to (Y, d_Y)$ be a continuous mapping. Then show that the space f(X) with the metric induced from Y is connected. (6)

(c) Prove that the intervals are connected sets in (R,d), the metric space of real numbers with the usual metric. (6)

6. (a) Let (X, d) be a metric space. Then prove that the following statements are equivalent:

(i) (X, d) is compact;

(ii) every collection of closed sets in (X, d) with empty intersection has a finite subcollection with empty intersection;

(iii) every collection of closed sets in (X, d) with the finite intersection property has nonempty intersection. (6.5)

(c) Let T be a continuous mapping of a complete metric space into itself. Define T^n inductively by $T^1 = T$ and $T^{n+1} = T \circ T^n$. If T^k is a contraction for some positive integer k, then the equation $Tx = x$ has one and only one solution. Moreover, for any $x \in X$, the sequence $\{T^n x\}_{n \geq 1}$ converges to that solution. (6.5)

5. (a) Let (X,d) be a metric space. Then show that the following statements are equivalent:

(i) (X,d) is disconnected;

(ii) There exist two nonempty disjoint subsets A and B, both open in X, such that $X = A \cup B$;

(iii) There exist two nonempty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(iv) There exists a proper subset of X that is both open and closed in X. (6)

(c) Suppose $X = C_R[a, b]$, the set of all continuous functions from [a,b] to $\mathbb{R}$ and $d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]; a<b\}$ be the associated metric. Then prove that (X, d) is a complete metric space. (6)

2. (a) (i) Suppose (X, d) is a metric space and F is a finite subset of X. Prove that the derived set F' of F is empty.

(ii) Let A be a finite subset of $\mathbb{R}$ with usual metric. Is A open in $\mathbb{R}$? Justify your answer. If not, then give an example of a metric space in which a finite set may be open. (2.5+4)

(b) (i) Consider the subspace $X = I \cup J$ of R, where $I =]0,1[$ and $J = [4,7[$. Show that $d(\sup_x I, J) > 0$.

(ii) Find the interior and derived sets of Z and Q in R with usual metric. Determine which one of these sets is open or closed or none in R. Justify your answer. (2.5+4)

(c) (i) Is Q $\cup\{\sqrt{2}\}$ dense in R with usual metric? Justify your answer.

(ii) Give an example of a set which is open in a subspace Y of a metric space (X,d) but not open in X. Under what condition every open set in a subspace Y of a metric space (X,d) is open in X? Justify your answer.

(2+4.5)

3. (a) Prove that a mapping $f: X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. (6)

(b) Define a uniformly continuous function on a metric space. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous, whereas the function $f: [0, 1] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous. (6)

(c) Let f be an isometry between metric spaces (X_1, d_1) and (X_2, d_2). Show that f is a homeomorphism between X_1 and X_2. Is the converse true? Justify your answer. (6)

4. (a) Show that the metrics d_1, d_2 and d_∞ defined on $\mathbb{R}^n$ by

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| ;$$

$$d_2(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}; \text{ and}$$

$$d_\infty(x,y) = max\,\{|x_i - y_i| : 1 \le i \le n\}$$

are equivalent. (6.5)

(b) Let $X = [0, 2\pi] \subset R$ and $Y = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle in the complex plane. Consider the map $f : X \to Y$ defined by $f(t) = e^{it}$, $0 \le t \le 2\pi$. Show that the function f is one-to-one, onto, continuous but not a homeomorphism. (6.5)