(b) Determine all group homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}_n$.

(c) Let H be a subgroup of G and a,b $\in$ G. Prove that

(i) $aH = bH \Leftrightarrow a \in bH$.

(ii) $aH = H \Leftrightarrow a \in H$. (2x6=12)

6 (a) State and prove the First Fundamental Theorem of Group Homomorphisms.

(b) Let G be the symmetric group of permutations on the set {1,2,3..., n}. For each σ in G, define

$$\mathrm{Sgn}(\sigma) = \begin{cases} -1, & \text{if } \sigma \text{ is an odd permutation} \\ 1, & \text{if } \sigma \text{ is an even permutation.} \end{cases}$$

Prove that Sgn is a group homomorphism. Find the kernel of Sgn.

(c) (i) Let $\phi: G \to G^*$ be a group homomorphism. Prove that the Ker & is a normal subgroup of the group G.

(ii) Show that : $\phi: (\mathbb{R},+) \longrightarrow (\mathbb{R}^*,.)$ defined as $\phi(x) = x^2$ is not a group homomorphism. (2×6½=13)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5904 **I**

Unique Paper Code : 32351302

Name of the Paper : BMATH306 - Group Theory-1

Name of the Course : **B.Sc. Maths (Hons) (2019 onwards) LOCF**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.
2. Attempt all questions by selecting two parts from each question.
3. All parts of a question are to be attempted together.
4. Marks of the questions are indicated.
5. Use of Calculator not allowed.

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1. (a) Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \sim \{0\} \right\}$. Show that G is an Abelian group.

P.T.O.

(b) Let G be group and H be a non-empty subset of G. Prove that H is a subgroup of G if

$ab^{-1} \in H, \forall a, b \in H.$

Hence prove that $H = \{A \in GL(2, \mathbb{R})$: det A is a power of 2$\}$ is a subgroup of GL(2, $\mathbb{R}$).

(c) Define a cyclic group. Show that every cyclic group is Abelian. Is the converse true? Justify. (2×6=12)

2. (a) Prove that an Abelian group with two elements of order two must have a subgroup of order 4.

(b) Describe the group of symmetries of an equilateral triangle.

(c) Let G be a group and $a \in G$. If $|a| = n$, show that

$< a >= \{e, a, a^2, \ldots\ldots, a^{n-1}\}$ & $a^i = a^j \Leftrightarrow$ n divides $(i - j)$. (2×6½=13)

3. (a) Let G = $<a>$ be a cyclic group of order n. Then show that G = $<a^k> \Leftrightarrow$ gcd (k, n) = 1.

(b) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 & 8 \end{pmatrix}$ and

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$

(i) Write α and β as product of disjoint cycles.

(ii) Compute $\alpha\beta$ and $\alpha^{-}1$.

(c) Suppose that a has order 15. Compute all the left cosets of $< a^5 >$ in $<a >$. (2x6=12)

4. (a) Define left coset of a subgroup H of a group G. Find all the left cosets of H={1,11} in U(30).

(b) (i) Prove that a subgroup H of a group G is normal in G $\Leftrightarrow xHx^{-1} \subseteq H, \forall x \in G$.

(ii) Let H be a subgroup of a group G such that $x^2 \in H, \forall x \in G$. Show that H is a normal subgroup of G.

(c) Leto: $\phi: G \longrightarrow G'$ be a group homomorphism and let H be a subgroup of G. Prove that

(i) if H is cyclic, then ϕ(H) is cyclic.

(ii) if H is normal, then ϕ(H) is normal in ϕ(G). (2×6½=13)

5. (a) Let G be a finite group and H be a subgroup of G. Show that |H| divides |G|.