

(c) Show that the improper integral $\int_1^{\infty} \frac{\sin x}{x^2} dx$

converges absolutely.

(d) Define the Gamma function $\Gamma(m)$. Prove that $\Gamma(m)$ converges if $m > 0$.

[This question paper contains 8 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 1140

I

Unique Paper Code : 2352012302

Name of the Paper : Riemann Integration

Name of the Course : **B.Sc. (H) Mathematics**
UGCF-2022

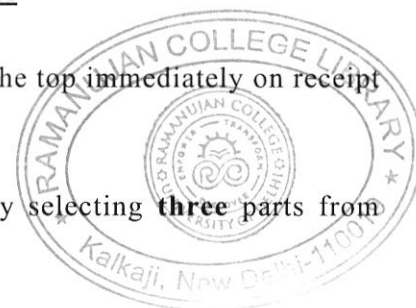
Semester : III DSCC-8

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator is not allowed.



1. (a) Find the upper and lower Darboux integrals for $f(x) = x^2$ on the interval $[0, b]$ and show that

$$\int_0^b x^2 = \frac{b^3}{3}.$$

- (b) Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and $P \subseteq Q$, then prove that

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

- (c) Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function on $[a, b]$. Prove that if f is integrable on $[a, b]$, then for each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

- (d) Let $f(x) = 2x + 1$ over the interval $[0, 2]$. Let

$$P = \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \right\} \text{ be a partition of } [0, 2]. \text{ Compute}$$

$$U(f, P), L(f, P) \text{ and } U(f, P) - L(f, P).$$

- (c) Find the exact arc length of the curve $x =$

$$\frac{1}{3}(y^2 + 2)^{3/2} \text{ from } y = 0 \text{ to } y = 1.$$

- (d) Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y -axis.

6. (a) Discuss the convergence or divergence of the following improper integrals :

$$(i) \int_0^1 \frac{1}{\sqrt{x}} dx$$

$$(ii) \int_{-\infty}^{+\infty} e^x dx$$

- (b) Find the value of r for which the integral $\int_0^1 x^{-r} dx$ exists or converges, and determine the value of the integral.

(i) Determine the function $F(x) = \int_0^x f(t) dt$.

(ii) Where is F continuous?

(d) For $t \in [0,1]$, define $F(t) = \begin{cases} 0, & t < \frac{1}{2} \\ 1, & t \geq \frac{1}{2} \end{cases}$ and let

$f(x) = x^2$, $x \in [0,1]$. Show that f is F -integrable

and that $\int_0^1 f dF = f\left(\frac{1}{2}\right)$.

5. (a) Find the volume of the solid generated when the region enclosed by the curves $x = \sqrt{y}$ and $x = y/4$ is revolved about the x -axis.

(b) Use cylindrical shells to find the volume of the solid generated when the region under $y = x^2$ is revolved about the line $y = -1$.

2. (a) Let f be an integrable function on $[a, b]$. Show that $-f$ is integrable on $[a, b]$ and

$$\int_a^b (-f) = -\int_a^b f$$

(b) Let $f: [0,2] \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

Calculate the upper and lower Darboux Integrals for f on the interval $[0,2]$. Is f integrable on $[0,2]$?

(c) Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. Show that if f is integrable (Darboux) on $[a, b]$, then it is Riemann integrable on $[a, b]$.

(d) For a bounded function f on $[a, b]$, define the Riemann Sum associated with a partition P . Hence, give Riemann's definition of integrability.

3. (a) Prove that every bounded piecewise monotonic function f on $[a, b]$ is integrable.

(b) Show that if a function f is integrable on $[a, b]$,

then $|f|$ is integrable on $[a, b]$ and $\left| \int_a^b f \right| \leq \int_a^b |f|$.

(c) If f is a continuous, non-negative function on

$[a, b]$ and if $\int_a^b f = 0$, then prove that f is

identically 0 on $[a, b]$. Give an example of a discontinuous non-zero function f on $[0, 1]$ for

which $\int_0^1 f = 0$.

(d) State and prove Fundamental Theorem of Calculus I.

4. (a) If u and v are continuous functions on $[a, b]$ that are differentiable on (a, b) , and u' and v' are

integrable, prove that $\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a)$.

Hence evaluate $\int_0^\pi x \cos x$.

(b) Use the Fundamental Theorem of Calculus to

calculate $\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x e^{t^2} dt$.

(c) Let f be an integrable function on $[a, b]$. For x

in $[a, b]$, let $F(x) = \int_a^x f(t) dt$. Then show that F

is uniformly continuous on $[a, b]$. For $f(x) =$

$$\begin{cases} 0, & t < 0 \\ t, & 0 \leq t \leq 1, \\ 4, & t > 1 \end{cases}$$