(b) Let $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x)=\begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

Show that g is differentiable for all $x \in \mathbb{R}$.

Also, show that the derivative g' is not continuous at $x = 0$. (6)

(c) Suppose that f is continuous on a closed interval $I = [a, b]$, and that f has a derivative in the open interval (a, b). Prove that there exists at least one point c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

Suppose that $f: [0, 2] \to \mathbb{R}$ is continuous on $[0,2]$ and differentiable on $(0,2)$ and that $f(0) = 0$, $f(2) = 1$. Show that there exists $c_1 \in (0,2)$ such $f'(c_1) = 1/2$. (6)

6. (a) Find the points of relative extrema of the function $f(x) = 1 - (x - 1)^{2/3}$, for $0 \leq x \leq 2$. (6)

(b) Let I be an open interval and let $f: I \to \mathbb{R}$ has a second derivative on I. Then show that f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$. (6)

(c) Obtain Taylor's series expansion for the function $f(x) = \sin x$, $\forall x \leq \mathbb{R}$. (6)

(1000)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : **4350** **G**

Unique Paper Code : 32351301

Name of the Paper : Theory of Real Functions

Name of the Course : **B.Sc. (H) Mathematics (LOCF)**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Ramanujan College Library, Kalkaji, New Delhi-110019

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A and $f: A \to \mathbb{R}$, then define limit of function f at c. Use $\in - \delta$ definition to show that $\lim\limits_{x \to 2} \frac{1}{1-x} = -1$. (6)

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x)=\begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

Show that f has a limit at $x = 0$. Use sequential criterion to show that f does not have a limit at c if $c \neq 0$. (6)

(c) Show that $\lim_{x \to 0} \cos\left(\frac{1}{x^2}\right)$ does not exist in $\mathbb{R}$ but $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$. (6)

2. (a) Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Show that if

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then $\lim_{x \to c} (fg)(x) = LM$. (6)

(b) Evaluate the limit $\lim_{x \to 1+} \frac{x}{x-1}$. (6)

(c) Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$. Show that the following conditions are equivalent-

(i) f is continuous at c.

(ii) For every sequence $\langle x_n \rangle$ in A that converges to c, the sequence $\langle f(x_n) \rangle$ converges to f(c). (6)

3. (a) Let $A, B \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ and $g = B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. if f is continuous at a point $c \in A$ and g is continuous at $b \in B$, then show that the composition function $g \circ f: A \to \mathbb{R}$ is continuous at c. Also, show that the function $f(x) = \cos(1 + x^2)$ is continuous on $\mathbb{R}$. (7½)

(b) State and prove Maximum-Minimum Theorem for continuous functions on a closed and bounded interval. (7½)

(c) State Bolzano's Intermediate value theorem. Show that every polynomial of odd degree with real coefficients has at least one real root. (7½)

4. (a) Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. Show that if f is continuous at $c \in A$ then $|f|$ is continuous at c. Is the converse true? Justify your answer. (6)

(b) Let $I = [a, b]$ and $f: I \to \mathbb{R}$. Show that if f is continuous on I then it is uniformly continuous on I. (6)

(c) Show that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$ and the function $g(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$ is not uniformly continuous on $(0, \infty)$. (6)

5. (a) Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be functions that are differentiable at c. Prove that the function fg is differentiable at c, and $(fg)' = f'(c)g(c) + f(c)g'(c)$. (6)