(c) (i) Show that $\left|\int_{-2\pi}^{2\pi} x^2 cos^8(e^x)\, dx\right| \leq \frac{16\pi^3}{3}$.

(ii) Give an example of a function f on [0, 1] that is not integrable for which $|f|$ is integrable on [0, 1].

(d) Suppose that f and g are continuous functions on $[a,b]$ such that $\int_a^b f\, dx = \int_a^b g\, dx$. Prove that there exists x in $[a,b]$ such that $f(x) = g(x)$.

4. (a) If f and g are two integrable functions on $[a,b]$, then prove that $(f+g)$ is also integrable on $[a,b]$.

(b) Prove that every piecewise monotonically increasing function on $[a,b]$ is integrable on $[a,b]$.

(c) State Fundamental Theorem of Calculus-II. Hence or otherwise evaluate $\lim_{h\to 0}\frac{1}{h}\int_4^{4+h} e^{t^2}\, dt$.

(d) Let f be defined on $\mathbb{R}$ as

$$f(t) = \begin{cases} t & for\ t < 0 \\ t^2+1 & for\ 0 \leq t \leq 2 \\ 0 & for\ t > 2. \end{cases}$$

Determine the function $F(x) = \int_0^x f(t)\, dt$.

(i) At what points F is continuous?

(ii) At what points F is differentiable? Calculate F' at the points of differentiability.

5. (a) Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval [0, 2] is rotated about the line $y = -1$.

(b) Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 4$, $x = 9$ and the x-axis is revolved about the y-axis.

(c) Find the exact arc length of the curve $y = x^{\frac{2}{3}}$ from $x = 1$ to $x = 8$.

(d) The circle $x^2 + y^2 = r^2$ is rotated about the x-axis to obtain a sphere. Find the surface area of the sphere.

6. (a) Discuss the convergence of following improper integrals:

(i) $\int_0^1 \frac{1}{x\ln x}\, dx$ (ii) $\int_1^\infty \frac{dx}{\sqrt{x^3+x}}$.

(b) Find a function f such that $\int_1^\infty f$ converges, but $\int_1^\infty \sqrt{f}$ does not converge. Justify your answer.

(c) Show that the improper integral $\int_0^1 t^{p-1}(1-t)^{q-1}\, dt$ converges if and only if p and q are positive.

(d) Show that

(i) $\Gamma(p+1) = p\,\Gamma(P)$ for all $p > 0$,

(ii) $\Gamma(n) = (n-1)!$ for all $n \in \mathbf{N}$.

Sr. No. of Question Paper	:	**1572**
Unique Paper Code	:	2352012302
Name of the Paper	:	DSC-8 : Riemann Integration
Programme	:	B.Sc. (Hons.) Mathematics (NEP-UGCF 2022)
Semester	:	III

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory. Attempt any **Three** parts from each question.
3. All questions carry equal marks.

1. (a) Let $f: [-1,1] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 2, & if\ x \in \mathbb{Q} \\ 3, & if\ x \notin \mathbb{Q} \end{cases}$$

Show that f is not integrable on [-1,1].

(b) Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Show that if f is integrable on $[a, b]$, then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $U(f, P) - L(f, P) < \varepsilon$ for every partition P of $[a, b]$ with mesh$(P) < \delta$.

(c) Let $f(x) = 3x + 2$ over the interval [1,3]. Let P be a partition of [1,3] given by $P = \{1, 3/2, 2, 3\}$. Compute $L(f, P), U(f, P)$ and $U(f, P) - L(f, P)$.

(d) Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Show that if P and Q are any partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$. Hence show that $L(f) \leq U(f)$.

(e)

2. (a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n\in\mathbb{N}}$ of $[a, b]$, satisfying $\lim[U(f, P_n) - L(f, P_n)] = 0$.

(b) Suppose that a function f defined on $[a, b]$ is integrable on $[a, c]$ and $[c, b]$, where $c \in (a, b)$. Prove that f is integrable on $[a, b]$ and that $\int_a^b f = \int_a^c f + \int_c^b f$.

(c) Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Show that if f is Riemann integrable on $[a, b]$, then it is (Darboux) integrable on $[a, b]$, and that the values of the integrals agree.

(d) For $t \in [0,1]$, let $F(t) = \begin{cases} 0 & \text{for } t < 1/3 \\ 1 & \text{for } t \geq 1/3. \end{cases}$

Let $f(x) = x^2$, where $x \in [0,1]$. Show that f is F-integrable and that $\int_0^1 f\, dF = f(1/3)$.

3. (a) Prove that every continuous function on $[a, b]$ is integrable on $[a, b]$.

(b) State and prove the Intermediate Value Theorem for Integrals.