5. (a) Let G be a group acting on a non-empty set A. If $a, b \in A$ and $b = g \cdot a$ for some $g \in G$. Prove that $G_b = g\, G_a\, g^{-1}$ where G_a is stabilizer of a in G. Deduce that if G acts transitively on A then kernel of action is $\cap_{g \in G}\, g\, G_a\, g^{-1}$.

(b) Define the action of a group G on itself by conjugation. Prove it is a group action. Also find the kernel of this action.

(c) If G is a group of order pq, where p and q are primes, $p < q$, and p does not divide q-1, then prove that G is cyclic.

6. (a) State the Class Equation for a finite group G. Find all the conjugacy classes for quaternion group Q_8 and also, compute their sizes. Hence or otherwise, verify the class equation for Q_8.

(b) Use Sylow theorems to determine if a group of order 105 is not simple.

(c) State and prove Embedding theorem and use it to prove that a group of order 112 is not simple.

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : **4386** **G**

Unique Paper Code : 32351502

Name of the Paper : Group Theory – II

Name of the Course : **B.Sc. (H) Mathematics**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Question No. **1** has been divided into **10** parts and each part is of **1.5** marks.
4. Each question from Q. Nos. **2** to **6** has **3** parts and each part is of **6** marks. Attempt any **two** parts from each question.

1. State true (T) or false (F). Justify your answer in brief.

(i) Let G be a finite group of order 147 then it has a subgroup of order 49.

(ii) There is a simple group of order 102.

P.T.O.

(iii) Dihedral Group D_{12} (having 24 elements) is isomorphic to the symmetric group S_4.

(iv) The action $z \cdot a = z + a$ of the additive group of integers Z on itself is faithful.

(v) The external direct product $G \oplus H$ is cyclic if and only if groups G and H are cyclic.

(vi) Trivial action is always faithful.

(vii) The group of order 27 is abelian.

(viii) The external direct product $Z_2 \oplus Z_6$ is cyclic.

(ix) Every Sylow p-subgroup of a finite group has order some power of p.

(x) A p-group is a group with property that it has atleast one element of order p.

2. (a) Prove that for every positive integer n, Aut $(Z_n) \cong U(n)$.

(b) Define Automorphism Aut(G) of a group G and Inner Automorphism Inn(G) of the group G induced by an element 'a' of G. Prove that $Aut(Z_5)$ is isomorphic to U(5), where $U(5) = \{1, 2, 3, 4\}$ is group under the multiplication modulo 5.

(c) Define characteristic subgroup of G. Prove that every subgroup of a cyclic group is characteristic.

3. (a) Prove that the order of an element of a direct product of finite number of finite groups is the least common multiple of the orders of the components in the elements. Find the largest possible order of an element in $Z_{30} \oplus Z_{20}$.

(b) Prove that if a group G is the internal direct product of finite number of subgroups $H_1, H_2, \ldots H_n$ then G is isomorphic to $H_1 \oplus H_2 \oplus H_3 \ldots \oplus H_n$.

(c) Let G is an abelian group of order 120 and G has exactly three elements of order 2. Determine the isomorphism class of G.

4. (a) Show that the additive group R acts on x,y plane $R \times R$ by $r.(x,y)=(x+ry,y)$.

(b) Let G be a group acting on a non-empty set A. Define

(i) kernel of group action

(ii) Stabilizer of a in G, for $a \in A$

(iii) Prove that kernel is a normal subgroup of G.

(c) Define the permutation representation associated with action of a group on a set. Prove that the kernel of an action of group G on a set A is the same as the kernel of the corresponding permutation representation of the action.