(c) Prove that for every continuous function f on [0, 1], the sequence of polynomials $B_n f \to f$ uniformly on [0, 1], where $(B_n f)$ is the sequence of Bernstein's polynomials for the function f.

[This question paper contains 8 printed pages.]

28/5/24 (M) **Your Roll No..............**

Sr. No. of Question Paper : 4064 **H**

Unique Paper Code : 2352012401

Name of the Paper : Sequence and Series of Functions

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions are compulsory and carry equal marks.

1. (a) Define uniform convergence of a sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$. If $A \subseteq \mathbb{R}$ and $\varphi: A \to \mathbb{R}$ then define the uniform norm of

φ on A. Discuss the uniform and pointwise convergence of the sequence (f_n), where

$$f_n(x) = \frac{x}{n} \text{ for } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

(b) Let (f_n) be the sequence of functions defined by

$$f_n(x) = \frac{1}{1+x^n} \quad \forall x \in [0,1],\ n \in \mathbb{N}.$$

Find the pointwise limit of the sequence (f_n). Does (f_n) converges uniformly? Justify your answer.

(c) Show that if (f_n) and (g_n) are two sequences of bounded functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ that converge uniformly to f and g respectively then prove that the product sequence $(f_n g_n)$ converge uniformly on A to fg. Give an example to show that in general the product of two uniformly convergent sequence may not be uniformly convergent.

6. (a) For cosine function C(x) and sine function S(x) prove the following :

(i) If $f: \mathbb{R} \to \mathbb{R}$ is such that $f''(x) = -f(x)$ for $x \in \mathbb{R}$ then there exist real numbers α and β such that $f(x) = \alpha C(x) + \beta S(x)$ for $x \in \mathbb{R}$.

(ii) If $x \in \mathbb{R}$, $x \geq 0$ then

$$1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

(b) State Abel's Theorem. Show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \text{.............} \quad -1 \leq x \leq 1$$

and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{.............}$.

5. (a) Define the radius of convergence and interval of convergence of power series. Check the uniform convergence of the following power series on [−1, 1]

$$x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+\ldots\ldots\ldots\ldots$$

(b) State and Prove Cauchy-Hadamard Theorem.

(c) Suppose that $f(x)=\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R>0$. Then prove that the series

$\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ has also radius of convergence R

and for $|x|<R$

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}.$$

2. (a) Let (f_n) be a sequence of integrable functions on [a, b] and suppose that (f_n) converges uniformly to f on [a, b]. Show that f is integrable on [a, b] and

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$$

(b) Let $f_n(x) = \frac{x^n}{n}$ for $x \in [0, 1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on [0, 1] and that the sequence (f_n') converges on [0, 1] to a function g but the convergence is not uniform.

(c) Show that the sequence $\left(\frac{x^n}{1+x^n}\right)$ does not converge uniformly on [0,2].

3. (a) State and prove Weierstrass M-Test for uniform convergence of series of functions.

(b) Show that the series $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ is uniformly convergent on [–a, a], a > 0 but is not uniformly convergent on $\mathbb{R}$.

(c) Discuss the pointwise convergence of the series of functions $\sum_{n=1}^{\infty} \frac{x^n}{2+3x^n}$ for $x \geq 0$.

4. (a) Let f_n be continuous function on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in N$ and $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on D. Prove that f is continuous on D.

(b) Show that the series of functions $\sum_{n=1}^{\infty} \frac{\cos\left(x^2+1\right)}{n^3}$ converges uniformly on $\mathbb{R}$ to a continuous function.

(c) Given the Riemann integrable functions $f_n: \mathbb{R} \to \mathbb{R}$, $n \in N$, such that

$$f_n(x) = \sin\left(\frac{x}{n^4}\right) \text{ for all } x \in \mathbb{R}$$

Show that $\sum_{n=1}^{\infty} \int_{-2\pi}^{2\pi} f_n(x)\, dx = \int_{-2\pi}^{2\pi} \sum_{n=1}^{\infty} f_n(x)\, dx$.