(1,1)} use the least square approximation to find the line of best fit.

(ii) For $V = \mathbb{R}^2$ with the standard inner product and a linear operator $T: V \to V$, given by $T(a, b) = (2a + b, a - 3b)$ for all $a, b \in V$, evaluate $T^*(3,5)$.

(4+2.5,3+3.5,4+2.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 2987 **H**

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra-II

Name of the Course : **B.Sc. (H) Mathematics (CBCS-LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** the questions are compulsory.
3. Attempt any **two** parts from each question.
4. Marks of each part are indicated.

1. (a) State and prove the Division Algorithm for $\mathbb{F}[x]$, where $\mathbb{F}$ is a field.

(b) State and prove Eisenstein's Criterion.

(c) Let $\mathbb{F}$ be a field and

$I = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n | a_i \in \mathbb{F},$

$i = 0,1,2,\ldots, n$ and $\sum_{i=0}^{n} a_i = 0\}$. Show that I is an ideal of $\mathbb{F}[x]$ and find a generator of I.

(6,6,6)

2. (a) Let $\mathbb{F}$ be a field and $p(x) \in \mathbb{F}[x]$. Prove that $<p(x)>$, is a maximal ideal in $\mathbb{F}[x]$ if and only if $p(x)$ is irreducible.

(b) Prove that every Euclidean Domain is a Principal Ideal Domain.

(c) In the ring $\mathbb{Z}[\sqrt{5}]$, show that the element $1+\sqrt{5}$ is irreducible but not prime. (6.5,6.5,6.5)

(i) T is self-adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in V$.

(ii) If $\langle Tx, x\rangle = 0$ for all $x \in V$, then $T = T_0$, the zero operator on V.

(b) (i) Show that the reflection operator on $\mathbb{R}^2$ about a line through the origin is an orthogonal operator.

(ii) Show that the pair of matrices $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$ are unitarily equivalent.

(c) (i) For the set of data $S = \{(-3,9), (-2,6), (0,2),$

$<f, g> = \int_0^{\pi} f(t)g(t)dt$ to obtain an orthogonal basis for V. Then, normalize the vectors in this basis to obtain an orthonormal basis β for V and compute the Fourier coefficients of the vector h(t) = 2t + 1 relative to β.

(c) Prove that an orthogonal subset of a finite-dimensional inner product space V can be extended to an orthonormal basis for V. Hence, or otherwise, prove that for any subspace W of a finite-dimensional inner product space V, $\dim(V) = \dim(W) + \dim(W^{\perp})$.

(4+2,6,6)

6. (a) Let T be a linear operator on a complex finite-dimensional inner product space V with an adjoint T*. Prove that

3. (a) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows :

$$f_1(x,y,z) = x - \frac{1}{2}y, \quad f_2(x,y,z) = \frac{1}{2}y, \quad \&$$

$$f_3(x,y,z) = -x + z$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V* and then find a basis for V for which it is the dual basis.

(b) Let $V = P_3(\mathbb{R})$ and $T(f(x)) = f(x) + f(2)x$. Show that T is a linear operator on V. Further, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) Test the linear operator $T: \mathbb{C}^2 \to \mathbb{C}^2$, $T(z, w) = (z + iw, iz + w)$ for diagonalizability and if diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix. (6,6,6)

4\. (a) Let T be a linear operator on a finite dimensional vector space V and let W be a T-invariant subspace of V. State relationship between the characteristic polynomial of T_W and characteristic polynomial of T. Verify that relationship for the linear operator $T: \mathbb{R}^4 \to R^4$, $T(a, b, c, d) = (a + b + 2c - d, b + d, 2c - d, c + d)$ and the T-invariant subspace $W = \{(t, s, 0, 0): t, s \in \mathbb{R}\}$ of V.

(b) State Cayley-Hamilton theorem for an n-dimensional vector space V and use it to prove Cayley-Hamilton theorem for matrices.

(c) Let T be a linear operator on a vector space Land let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1, 2, \ldots, k$, let S_i be a finite linearly independent subset of the eigenspace E_{λ_i}. Then prove that $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of V. (6.5,6.5,6.5)

5\. (a) (i) Let $V = M_{2\times2}(\mathbb{C})$ together with the Frobenius inner product given by

$\langle A, B\rangle = tr(B^*A)$ for all $A, B \in V$. Let

$A = \begin{bmatrix} 1 & 2+i \\ 3 & i \end{bmatrix}$ and $A = \begin{bmatrix} 1+i & 0 \\ i & -i \end{bmatrix}$. Compute

$\langle A, B\rangle$, $\|A\|$, $\|B\|$ and $\|A + B\|$. Then, verify both the Cauchy-Schwarz inequality and the triangle inequality.

(ii) Provide a reason why $\langle(a, b), (c, d)\rangle = ac - bd$ is not an inner product on $\mathbb{R}^2$.

(b) Apply the Gram-Schmidt process to a subset $S = \{\sin t, \cos t, 1, t\}$ of the inner product space $V = span(S)$ with the inner product

P.T.O.