(c) Let V and W be vector spaces. Let $T: V \to W$ be linear and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of Range of T. Prove that if $S = \{v_1, v_2, \ldots v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, \ldots, k$. Then S is linearly independent. (6.5)

6. (a) Let T be a linear operator on a finite dimensional vector space V. Let β and β' be the ordered basis for V. Suppose that Q is the change of coordinate matrix that changes β' coordinates into β coordinates, then $[T]_{\beta'} = Q^{-1}[T]_{\beta} Q$. (6.5)

(b) Let β, γ be the standard ordered basis of $P_1(\mathbb{R})$ and $\mathbb{R}^2$ respectively.

Let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ be a linear transformation defined by $T(a + bx) = (a, a+b)$.

Find $[T]_{\beta}^{\gamma}$, $\left[T^{-1}\right]_{\gamma}^{\beta}$ and verify that $\left[T^{-1}\right]_{\gamma}^{\beta} = \left([T]_{\beta}^{\gamma}\right)^{-1}$. (6.5)

(c) Let $U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformations respectively defined by $U(f(x)) = f'(x)$ and $T(f(x)) = \int_0^x f(t)\,dt$. Prove that $[UT]_{\beta} = [U]_{\alpha}^{\beta}[T]_{\beta}^{\alpha}$ where α and β are standard ordered basis of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. (6.5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 3116 **H**

Unique Paper Code : 32351403

Name of the Paper : Ring Theory & Linear Algebra-I

Name of the Course : **B.Sc. (Hons.) Mathematics CBCS (LOCF)**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.

1. (a) Prove that every finite Integral domain is a field. Give an example of an infinite integral domain which is not a field, Justify. (6)

(b) (i) Let F be a field of order 2^n. Prove that the characteristic of F is 2.

(ii) Find all the units in $\mathbb{Z}[i]$ (6)

(c) Prove that the set of all the nilpotent elements of a commutative ring form a subring. (6)

2. (a) Let R be a commutative ring with unity and A be an ideal of R then prove that R/A is an integral domain if and only if A is a prime ideal of R. (6)

(b) Prove that $\mathbb{Z}[i]/<1 - i>$ is a field. (6)

(c) Find all the maximal ideals of $\mathbb{Z}_{20}$. (6)

3. (a) Find all the ring homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_{10}$. (6.5)

(b) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a,b,c \in \mathbb{Z} \right\}$ and Φ be the mapping that takes $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ to a. Show that

(i) Φ is a ring homomorphism

(ii) Determine the kernel of Φ.

(iii) Is Φ a one-one mapping. Justify. (6.5)

(c) State and prove first isomorphism theorem for rings. (6.5)

4. (a) Let V(F) be a vector space.

(i) Prove that the intersection of two subspaces of V(F) is also a subspace of V(F).

(ii) Show that union of two subspaces of V(F) may not be a subspace of V(F). Discuss

the condition under which union of two subspaces will also form a subspace of V(F). (6)

(b) Let S be a linearly independent subset of a vector space V(F), and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent iff $v \in$ Span (S). (6)

(c) Let u, v, w be distinct vectors of a vector space V. Show that if {u,v,w} is a basis for V, then {u+v+w, v+w, w} is also a basis for V. (6)

5. (a) Let V(F) and W(F) be vector spaces and let $T: V \to W$ be a linear transformation. If V is a finite-dimensional, then

Dim (V) = Nullity (T) + Rank (T). (6.5)

(b) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $U: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformations respectively defined by

$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

$U(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$

Let β, γ be the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, Prove that

(i) $[T+U]_\beta^\gamma + [T]_\beta^\gamma + [U]_\beta^\gamma$

(ii) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a. (6.5)