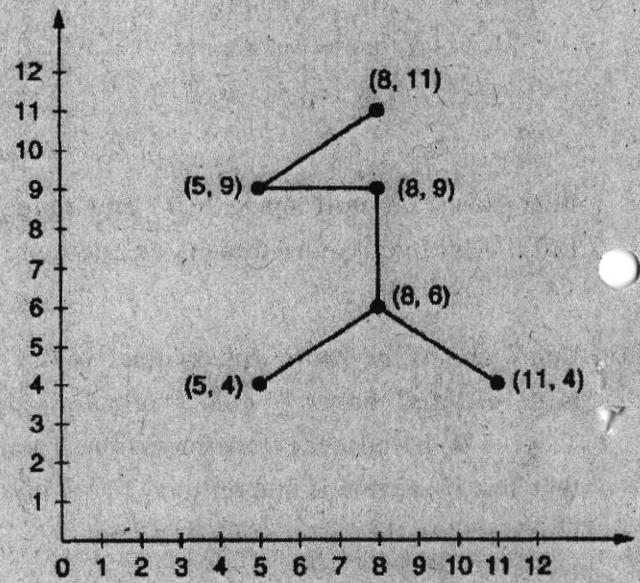

Also, sketch the final figure that would result from this movement.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4083 **H**

Unique Paper Code : 2352011201

Name of the Paper : Linear Algebra

Name of the Course : **B.Sc. (H) Mathematics – DSC**

Semester : II

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator not allowed.

1. (a) If x and y are vectors in $\mathbb{R}^n$, then prove that $|x \cdot y| \leq \|x\|\,\|y\|$. Also, verify the same for the vectors x = [4,2,0,–3,–1] and y = [1,4,–1,0,2] in $\mathbb{R}^5$.

(b) Using the Gauss – Jordan method, find the complete solution set for the following non-homogeneous system of linear equations :

$$2x_1 + 4x_2 - x_3 = 9$$

$$3x_1 - x_2 + 5x_3 = 5$$

$$8x_1 + 2x_2 + 9x_3 = 19$$

(c) Define the row space of a matrix. Also, determine whether the vector v = [7,1,18] is in the row space of the following matrix :

$$A = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 10 & -4 \\ 2 & -1 & 4 \end{pmatrix}$$

If so, then express [7,1,18] as a linear combination of the rows of A.

$$L\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 + u_3 \\ u_1 + u_2 \\ u_2 - u_3 \end{pmatrix}$$

Find bases for null space N(T) and range space R(T). Also, verify the dimension theorem.

6. (a) Let V and W be finite dimensional vector spaces with ordered basis β and γ respectively. Let T: V → W be a linear transformation. Then prove that T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

(b) Let V and W be finite dimensional vector spaces and let T: V → W be an isomorphism. Let V_0 be a subspace of V.

(i) Prove that $T(V_0)$ is a subspace of W.

(ii) Prove that $\dim(V_0) = \dim(T(V_0))$.

(c) For the adjoining graphic, use homogenous co-ordinates to find the new vertices after performing scaling about (8,4) with scale factors of 2 in the x-direction and 1/3 the y-direction.

(c) Let W be a subspace of a finite dimensional vector space V. Prove that W is finite dimensional and $\dim(W) \le \dim(V)$. Further, if $\dim(W) = \dim(V)$, then show that $V = W$.

5. (a) Let V and W be two finite dimensional vector spaces and let $T: V \to W$ be a linear transformation. Then prove that

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Also, prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a_1, a_2, a_3) = (-2a_1 + 3a_3, a_1 + 2a_2 - a_3)$ with $\beta = \{(1,-3,2), (-4,13,-3), (2,-3,20)\}$ and $\gamma = \{(-2,-1), ([illegible],3)\}$. Compute $[T]_\beta^\gamma$. Is T one to one?

(c) Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by :

2. (a) Define the rank of a matrix. Consider the matrix :

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 5 & -4 & 3 \end{pmatrix}$$

Using rank of A, determine whether the homogeneous system $AX = 0$ has a non-trivial solution or not.

(b) Consider the matrix :

$$A = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{pmatrix}.$$

(i) Find the eigenvalues and the fundamental eigenvectors of A.

(ii) Is A diagonalizable? Justify your answer.

(c) Find the quadratic equation of the form $y = ax^2 + bx + c$ that passes through the points $(-2,20)$, $(1,5)$, and $(3,25)$.

3. (a) (i) Let $\mathcal{C}$ be a family of subspaces of a vector space V and let W denote the intersection of subspaces in $\mathcal{C}$. Prove that W is a subspace of V.

(ii) Let F be any field. Prove that $W = \{(a_1, a_2, \ldots, a_n) \in F^n \mid a_1 + a_2 + \cdots a_n = 0\}$ is a subspace of F^n.

(b) Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

(c) (i) Show that if S_1 and S_2 are arbitrary subsets of a vector space V_1 then

$$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2).$$

Give an example in which $\operatorname{span}(S_1 \cap S_2)$ and $\operatorname{span}(S_1) \cap \operatorname{span}(S_2)$ are unequal.

(ii) Does the polynomial $-x^3 + 2x^2 + 3x + 5$ belong to span (S), where

$$S = \{x^3 + x + 1,\ x^3 - 2x^2 + 1,\ x^2 + x + 1).$$

4. (a) Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Prove that $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$.

(b) Let $V = M_{2\times2}(F)$, $W_1 = \left\{\begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V \,\middle|\, a; b, c \in F\right\}$ and

$$W_2 = \left\{\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V \,\middle|\, a, b \in F\right\}.$$

Find a basis for the subspaces W_1, W_2 and $W_1 \cap W_2$. Also, find the dimension of each of them.