7. (a) For the (M/M/1): (FIFO/N/∞) queuing model, derive the steady state equations and find the probability distribution of number of customers in the system. Hence obtain the expected queue length.

(b) Customers arrive at a booking counter being managed by a single cashier according to Poisson input process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with mean of 90 seconds. Determine the

(i) average number of customers in the system;

(ii) average number of customers in the queue; and

(iii) average waiting time of a customer in the queue. (7,8)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1017 C

Unique Paper Code : 32371501

Name of the Paper : Stochastic Processes and Queuing Theory

Name of the Course : B.Sc. (Hons.) STATISTICS: CBCS

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **five** questions in all.
3. **Section I** is compulsory.
4. Attempt **four** more questions, selecting **two** questions from each of **Section II** and **III**.
5. Use of non-programmable simple scientific calculator is allowed.

SECTION I

1. Attempt any **five** parts.

(i) In a classical ruin problem show that a fair game always remains fair and no unfair game can be changed to a fair game.

(ii) Obtain the distribution of the time interval between the two consecutive arrivals if the number of arrivals in any time interval follows a Poisson distribution.

(iii) Let p_n be the probability that n Bernoulli trials result in an even number of successes. Find the generating function of $\{p_n\}$. Hence obtain an asymptotic value of p_n using Partial Fraction Theorem.

(iv) Consider a Poisson process with parameter 5. Given that 15 occurrences have occurred by time 10, what is the expected number of occurrences (rounded off to the nearest integer) by time 8.

(v) Let $\{X_n, n \geq 1\}$ be a sequence of uncorrelated random variables with mean 0 and variance 1. Is $\{X_n, n \geq 1\}$ covariance stationary?

6. (a) Patients arrive at a lab to get tested for cervical cancer at a rate of 50 per day in accordance with a Poisson process. First an X-ray is conducted and 50% patients are found negative. The rest are tested further and a CT scan is done. On average 40% patients are found negative at this stage and the rest have to pass through a MRI test. 20% patients are found to be positive by MRI test which have to go through the treatment if cervical cancer. However the success rate of treatment is 50%. Among these 10% show a relapse within a year. What is the expected number of relapsed cases per year? Relapsed cases have a very high probability of death within the year of relapse. Estimate the expected number of deaths due to the disease reported by the lab.

(b) Let the footfall at a departmental store be described by a Poisson process at a rate of 130 per hour. There are three exits A, B and C from each of which the customers depart with equal probability; departures occurring again according to a Poisson process at a rate of 100 per hour. Find the probability of 5 arrivals between 50^{th} and 52^{nd} departure. Also find the mean number of departures from exit A between 10^{th} and 11^{th} arrivals. (7,8)

SECTION III

5. (a) A bacterial disease intensifies with division of bacterial cells which occur at rate λ per minute. Simultaneously bacterial cells die at a rate of μ per minute. The division and death rates are directly proportional to initial number n of bacteria in the colony. A new drug has been invented which will enhance the death rate by a rate 'a' per minute, which is constant, irrespective of the number of bacteria, along with the natural death mechanism of bacterial cells. Find an expression for the mean number of bacterial cells at time t in the body of the patient.

(b) In a classical ruin problem let q_z denote the probability of gambler's ruin and p_z denote the probability of gambler's winning the game. Show that $p_z + q_z = 1$ by obtaining an appropriate expression for p_z and also obtain the probability of gambler's ruin when gambler plays against an infinitely rich adversary. State all assumptions you make. (7,8)

(vi) Let $\{X_n, n \geq 0\}$ be a Markov chain having state space S = {0,1,2} and transition probability matrix as

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

The initial distribution is $P[X_0 = i] = 1/3, i = 0,1,2$.

Find $P[X_3 = 2, X_2 = 1, X_1 = 1, X_1 = 0]$.

(vii) A birth process has a p.g.f. G(s, t) given by $G(s,t) = \frac{s}{e^{\lambda t} + s(1 - e^{\lambda t})}$. Find the variance of the population size at time t. (3×5)

SECTION II

2. (a) In a random sample of school children, children are examined for any genetic abnormalities and only those who have at least one abnormality are recorded. The maximum number of abnormalities is n. Any abnormality occurs with constant probability p, irrespective of its type. Let X_i be a random variable denoting the number of abnormalities in the i^{th} recorded children. Obtain

the p.g.f of X_i. Hence obtain the expected number of abnormalities along with its variance in any of the recorded children.

(b) Three friends Dhruv, Akash, and Priya send SMS from one to another. Dhruv sends SMS to Akash, with probability $\frac{2}{3}$ and Akash sends SMS to the other two with equal probability. Priya sends SMS to Dhruv only.

(i) Present the information by a Markov chain.

(ii) In the long run, how often does each receive an SMS? (7,8)

3. (a) Let $\{X_n, n \geq 0\}$ be a Markov chain having state space $S = \{0,1,2,3\}$ and transition probability matrix as

$$P = \begin{pmatrix} 0.2 & 0.3 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.2 & 0.5 & 0 \\ 0 & 0.2 & 0.3 & 0.5 \end{pmatrix}$$

(i) Is the chain irreducible?

(ii) Find the nature of states 0,1,2 and 3.

(b) Let $S_N = X_1 + X_2 + \ldots + X_N$, where X_i's are i.i.d. Binomial (n, p) random variables and N is a Poisson(λ) variable, distributed independently of X_i's. Show that .

$$Var(S_N) = E(N)Var(X_i) + Var(N)(E(X_i))^2$$

Show further that $\frac{Var(S_N)}{(E(S_N))^2} \geq k\frac{Var(X_i)}{(E(X_i))^2}$, where k is related to the compounding parameter.

(7,8)

4. (a) With usual notations of a Markov chain, show that if state k is persistent null, then for every j, $\lim_{n\to\infty} p_{jk}^{(n)} \to 0$ and if state k is aperiodic, persistent non-null, then $\lim_{n\to\infty} p_{jk}^{(n)} \to \frac{F_{jk}}{\mu_{kk}}$, where symbols have their usual meaning.

(b) Define a covariance stationary stochastic process and a strictly stationary process. Let $\{Y_n = a_0X_n + a_1X_{n-1}\}$, n = 1,2..., where a_0, a_1 are constants and X_n, n = 0,1,2... are i.i.d. random variables with mean 0 and variance σ^2. Is $\{Y_n, n \geq 1\}$ covariance stationary? (7,8)