

- (i)  $G$  acts transitively on  $A$
- (ii) The stabilizer of the point  $1H \in A$  is the subgroup  $H$ .
- (iii)  $\text{Ker } \pi_H = \bigcap_{x \in G} xHx^{-1}$
5. (a) Let  $G$  be a permutation group on a set  $A$  ( $G$  is subgroup of  $S_A$ ), let  $\sigma \in G$  and let  $a \in A$ . Prove that  $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$ , here  $G_x$  denotes stabilizer of  $x$ . Deduce that if  $G$  acts transitively on  $A$  then  $\bigcap_{x \in G} \sigma G_a \sigma^{-1} = 1$ .
- (b) Show that every group of order 56 has a proper nontrivial normal subgroup.
- (c) State Index theorem and prove that a group of order 80 is not simple.
6. (a) State the Class Equation for a finite group  $G$ . and use it to prove that  $p$ -groups have non trivial centers.
- (b) Prove that group of order 255 is always cyclic.
- (c) Show that the alternating group  $A_5$  does not contain a subgroup of order 30, 20, or 15.

(1500)

[This question paper contains 4 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 1049

C

Unique Paper Code : 32351502

Name of the Paper : BMATH512: Group Theory-II

Name of the Course : B.Sc. (H) Mathematics

Semester : V

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

- Write your Roll No. on the top immediately on receipt of this question paper.
- All questions are compulsory.
- Question No. 1 has been divided into 10 parts and each part is of 1.5 marks.
- Each question from Q. Nos. 2 to 6 has 3 parts and each part is of 6 marks. Attempt any two parts from each Question.

- State true (T) or false (F). Justify your answer in brief.

(i) A group of order  $p^2$ ,  $p$  is a prime is always isomorphic to  $Z_{p^2}$ .

P.T.O.

- (ii) A group of order 15 is always cyclic.
- (iii) A group of order 14 is simple.
- (iv) The smallest positive integer  $n$  such that there are two non-isomorphic groups of order  $n$  is 6.
- (v) Every inner automorphism induced by an element 'a' of group  $G$  is an automorphism of  $G$ .
- (vi) A abelian group of order 12 must have an element of order either 2 or 3.
- (vii)  $U(105)$  is isomorphic to external direct product of  $U(21)$  and  $U(5)$ .
- (viii) Center of a group  $G$  is always a subgroup of normalizer of  $A$  in  $G$ , where  $A$  is any subset of  $G$ .
- (ix)  $\text{Aut}(Z_{10})$  is a cyclic group of automorphisms of  $G$ .
- (x) The largest possible order for an element of  $Z_{20} \oplus Z_{30}$  is 60.
2. (a) Define inner automorphism induced by an element 'a' of group  $G$  and find the group of all inner automorphisms of  $D_4$ .
- (b) Define the characteristic and the commutator subgroup of a group. Prove that the centre of a group is characteristic subgroup of the group.

- (c) Let  $G'$  be the subgroup of commutators of a group  $G$ . Prove that  $G/G'$  is abelian. Also, prove that if  $G/N$  is abelian, then  $N \geq G'$ .
3. (a) Determine the number of cyclic subgroups of order 15 in  $Z_{90} \oplus Z_{36}$ .
- (b) Define the internal direct product of the subgroups  $H$  and  $K$  of a group  $G$ . Prove that every group of order  $p^2$ , where  $p$  is a prime, is isomorphic to  $Z_{p^2}$  or  $Z_p \oplus Z_p$  (external direct product of  $Z_p$  with itself).
- (c) Consider the group  $G = \{1, 9, 16, 22, 29, 53, 74, 79, 81\}$  under multiplication modulo 91. Determine the isomorphism class of  $G$ .
4. (a) Show that the additive group  $Z$  acts on itself by  $z.a = z+a$  for all  $z, a \in Z$ .
- (b) Show that an action is faithful if and only if its kernel is the identity subgroup.
- (c) Let  $G$  be a group. Let  $H$  be a subgroup of  $G$ . Let  $G$  act by left multiplication on the set  $A$  of all left cosets of  $H$  in  $G$ . Let  $\pi_H$  be the permutation representation of  $G$  associated with this action. Prove that