(c) (i) For the inner product space $V = P_1(R)$ with $<f, g> = \int_{-1}^{1} f(t)g(t)dt$ and the linear operator T on V defined by $T(f) = f' + 3f$, compute $T^*(4 - 2t)$.

(ii) For the standard inner product space $V = \mathbb{R}^3$ and a linear transformation $g: V \to \mathbb{R}$ given by $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$, find a vector $y \in V$ such that $g(x) = <x, y>$ for all $x \in V$. (6,6,2+4)

6. (a) Prove that a normal operator T on a finite-dimensional complex inner product space V yields an orthonormal basis for V consisting of eigenvectors of T. Justify the validity of the conclusion of this result if V is a finite-dimensional real inner product space.

(b) Let $V = M_{2\times2}(\mathbb{R})$ and $T: V \to V$ be a linear operator given by $T(A) = A^T$. Determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(c) For the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$. (6.5,6.5,6.5)

26/5/23 [M]

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4792 E

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra – II

Name of the Course : **B.Sc. (H) Mathematics (CBCS – LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

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Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** the questions are compulsory.
3. Attempt any **two** parts from each question.
4. Marks of each part are indicated

1. (a) (i) Prove that If F is a field, then F[x] is a Principal Ideal Domain.

(ii) Is $\mathbb{Z}[x]$, a Principal Ideal Domain? Justify your answer.

(b) Prove that $<x^2 + 1>$ is not a maximal ideal in $\mathbb{Z}[x]$.

(c) State and prove the reducibility test for polynomials of degree 2 or 3. Does it fail in higher order polynomials? Justify. (4+2,6,6)

2. (a) (i) State and prove Gauss's Lemma.

(ii) Is every irreducible polynomial over $\mathbb{Z}$ primitive? Justify.

(b) Construct a field of order 25.

(c) In $\mathbb{Z}\left[\sqrt{(-5)}\right]$, prove that $1+3\sqrt{(-5)}$ is irreducible but not prime. (4+2.5,6.5,6.5)

3. (a) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows :

$f_1(x, y, z) = x - 2y$,
$f_2(x, y, z) = x + y + z$,
$f_3(x, y, z) = y - 3z$.

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^* and then find a basis for V for which it is the dual basis.

(b) Test the linear operator $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f(0) + f(1)(x + x^2)$ for diagonalizability and if diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) Let $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Find an expression for A^n where n is an arbitrary natural number. (6,6,6)

4. (a) For a linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(a, b, c) = (-b + c, a + c, 3c)$, determine the T-cyclic subspace W of $\mathbb{R}^3$ generated by $e_1 = (1, 0, 0)$. Also find the characteristic polynomial of the operator T_W.

(b) State Cayley-Hamilton theorem and verify it for the linear operator $T:P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f'(x)$.

(c) Show that the vector space $\mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$ where $W_1 = \{(a, b, 0, 0): a, b \in \mathbb{R}\}$, $W_2 = \{(0,0, c, 0): c \in \mathbb{R}\}$ and $W_3 = \{(0,0,0, d): d \in \mathbb{R}\}$. (6.5,6.5,6.5)

5. (a) Consider the vector space $\mathbb{C}$ over $\mathbb{R}$ with an inner product $\langle .,. \rangle$. Let $\bar{z}$ denote the conjugate of z. Show that $\langle .,. \rangle'$ defined by $\langle z, w\rangle' = \langle \bar{z}, \bar{w}\rangle$ for all $z, w \in \mathbb{C}$ is also an inner product on $\mathbb{C}$. Is $\langle .,. \rangle''$ defined by $\langle z, w\rangle'' = \langle z + \bar{z}, w + \bar{w}\rangle$ for all $z, w \in \mathbb{C}$ an inner product on $\mathbb{C}$? Justify your answer.

(b) Let $V = P(\mathbb{R})$ with the inner product $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(t)q(t)\,dt \;\; \forall p(x), q(x) \in V$. Compute the orthogonal projection of the vector $p(x) = x^{2k-1}$ on $P_2(\mathbb{R})$, where $k \in \mathbb{N}$.