(c) Let V, W and Z be finite dimensional vector spaces with ordered basis α, β, γ respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations.

Then $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$. (6.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4810 **E**

Unique Paper Code : 32351403

Name of the Paper : Ring Theory & Linear Algebra – I

Name of the Course : **B.Sc. [Hons.] Mathematics CBCS (LOCF)**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.

1. (a) Find all the zero divisors and units in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$. (6)

(b) Prove that characteristic of an integral domain is 0 or prime number p. (6)

(c) State and prove the Subring test (6)

2. (a) Let R be a commutative ring with unity and let A be an ideal of R then prove that R/A is a field if and only if A is a maximal ideal of R. (6)

(b) Let A and B are two ideals of a commutative ring R with unity and A+B=R then show that $A \cap B = AB$. (6)

6. (a) Let T be the linear operator on $\mathbb{R}^2$ define by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

Let β be the standard ordered basis for $\mathbb{R}^2$ and let

$$\beta' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}.$$

Find $[T]_{\beta'}$. (6.5)

(b) Let V and W be finite dimensional vector spaces with ordered basis β and γ respectively. Let $T: V \rightarrow W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

Furthermore, $\left[T^{-1}\right]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$. (6.5)

Find Null space and Range space of T and verify Dimension Theorem. (6.5)

(b) Define $T: M_{2X2}(\mathbb{R}) \to P_2(\mathbb{R})$ by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$

Let $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ and $\gamma = \{1, x, x^2\}$ be basis of $M_{2X2}(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. Compute $[T]_\beta^\gamma$. (6.5)

(c) Let V and W be vector spaces over F, and suppose that $\{v_1, v_2,\ldots, v_n\}$ be a basis for V. For $w_1, w_2,\ldots, w_n$ in W. Prove that there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$ for $i = 1,2,\ldots, n$. (6.5)

(c) If an ideal I of a ring R contains a unit then show that I=R. Hence prove that the only ideals of a field F are {0} and F itself. (6)

3. (a) Find all ring homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_{15}$. (6.5)

(b) Let $R = \left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{Z}\right\}$ and Φ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to a-b. Show that

(i) Φ is a ring homomorphism.

(ii) Determine Ker Φ.

(iii) Show that R/Ker Φ is isomorphic to $\mathbb{Z}$. (6.5)

(c) Using homomorphism, prove that an integer n with decimal representation $a_k a_{k-1} \dots \dots a_0$ is divisible by 9 iff $a_k + a_{k-1} + \dots + a_0$ is divisible by 9. (6.5)

4. (a) Let V(F) be the vector space of all real valued function over $\mathbb{R}$.

Let $V_e = \{f \in V \mid f(x) = f(-x) \ \forall \ x \in \mathbb{R})$

and $V_o = \{f \in V \mid f(-x) = -f(x) \ \forall \ x \in \mathbb{R})$

Prove that V_e and V_0 are subspaces of V and $V = V_e \oplus V_o$. (6)

(b) Let V(F) be a vector space and let $S_1 \subseteq S_2 \subseteq V$.

Prove that

(i) If S_1 is linearly dependent then S_2 is linearly dependent

(ii) If S_2 is linearly independent then S_1 is linearly independent (6)

(c) Show that $S = \left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right\}$ forms a basis for $M_{2X2}(\mathbb{R})$. (6)

5. (a) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$