[This question paper contains 2 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : **4686** **E**

Unique Paper Code : 32351402

Name of the Paper : Riemann Integration and Series of Functions

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **two** parts from each question.

1(a) Let f be integrable on [a, b], and suppose g is a function on [a, b] such that $g(x) = f(x)$ except for finitely many x in [a, b]. Show g is integrable and $\int_a^b g = \int_a^b f$ (6)

(b) Show that if f is integrable on [a, b] then f^2 also is integrable on [a, b]. (6)

(c) (i) Let f be a continuous function on [a,b] such that $f(x) \geq 0$ for all $x \in [a,b]$. Show that if $\int_a^b f(x)\,dx = 0$ then $f(x) = 0$ for all $x \in [a,b]$ (3)

(ii) Give an example of function such that $|f|$ is integrable on $[0,1]$ but f is not integrable on $[0,1]$. Justify it. (3)

2(a) State and prove Fundamental Theorem of Calculus I. (6.5)

(b) State Intermediate Value Theorem for Integrals. Evaluate $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$. (6.5)

(c) Let function $f: [0,1] \to R$ be defined as

$$f(x) = \begin{cases} x^2 & if\ x\ is\ rational \\ 0 & if\ x\ is\ irrational \end{cases}$$

Calculate the upper and lower Darboux integrals for f on the interval $[0,1]$. Is f integrable on $[0.1]$? (6.5)

3(a) Examine the convergence of the improper integral $\int_0^\infty e^{-x}x^{n-1}\,dx$. (6)

(b) Show that the improper integral $\int_\pi^\infty \frac{Sinx}{x}\,dx$ is convergent but not absolutely convergent. (6)

(c) Determine the convergence or divergence of the improper integral (6)

(i) $\int_0^1 \frac{dx}{x(lnx)^2}$

(ii) $\int_1^\infty \frac{xdx}{\sqrt{x^3+x}}$

4(a) Show that the sequence

$$f_n(x) = \frac{nx}{1+nx}, \quad x \in [0,1], \; n \in N$$

converges non-uniformly to an integrable function f on [0,1] such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$ (6.5)

(b) Show that the sequence $\{x^2 e^{-nx}\}$ converges uniformly on $[0, \infty)$. (6.5)

(c) Let $\langle f_n \rangle$ be a sequence of continuous function on $A \subset R$ and suppose that $\langle f_n \rangle$ converges uniformly on A to a function $f: A \longrightarrow R$. Show that f is continuous on A. (6.5)

5(a) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$. Show that sequence $\langle f_n \rangle$ converges non-uniformly on $[0, \infty)$ and converges uniformly on $[a, \infty), a > 0$. (6.5)

(b) State and prove Weierstrass M-test for the uniform Convergence of a series of functions. (6.5)

(c) Show that the series of functions $\sum \frac{1}{n^2+x^2}$, converges uniformly *on* R to a continuous function. (6.5)

6(a) (i) Find the exact interval of convergence of the power series (3)

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}$$

(ii) Define $sinx$ as a power series and find its radius of convergence (3)

(b) Prove that $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ for $|x| < 1$ and hence evaluate $\frac{\sum_{n=1}^{\infty} n^2(-1)^n}{3^n}$. (6)

(c) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad for \; |x| < R.$$ (6)