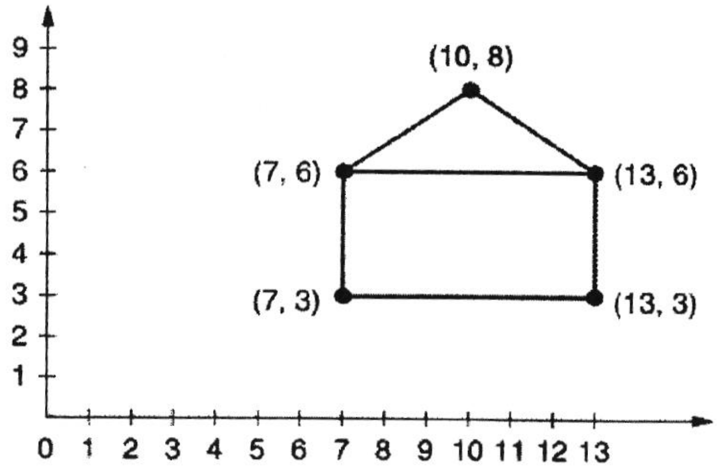

Also, sketch the final figure that would result from this movement.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1204 **F**

Unique Paper Code : 2352011201

Name of the Paper : Linear Algebra

Name of the Course : **B.Sc. (H) Mathematics**

Semester / Type : II / DSC

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator not allowed.

1. (a) If x and y are vectors in $\mathbb{R}^n$, then prove that $||x + y|| \leq ||x|| + ||y||$. Also, verily the same for the vectors $x = [-1, 4, 2, 0, -3]$ and $y = [2, 1, -4, -1, 0]$ in $\mathbb{R}^5$.

(b) Using the Gauss – Jordan method, find the complete solution set for the following homogeneous system of linear equations:

$$4x_1 - 8x_2 - 2x_3 = 0$$

$$3x_1 - 5x_2 - 2x_3 = 0$$

$$2x_1 - 8x_2 + x_3 = 0$$

(c) Define the rank of a matrix. Using rank, find whether the non-homogeneous linear system AX = B, where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 0 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

has a solution or not. If yes, find the solution.

2. (a) Consider the matrix :

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & -1 & 5 \\ -4 & -3 & 3 \end{pmatrix}$$

Determine whether the vector [4, 0, –3] is in the row space of A. If so, then express [4, 0, –3] as a linear combination of the rows of A.

6. (a) Let V and W be finite dimensional vector spaces over the same field F. Then, prove that V is isomorphic to W if and only if dim V = dim W. Are $M_{2\times2}(\mathbb{R})$ and $P_3(\mathbb{R})$ isomorphic? Justify your answer.

(b) Let V and W be vector spaces and let $T: V \to W$ be linear and invertible. Prove that $T^{-1}: W \to V$ is linear. For the linear transformation $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by :

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$$

determine whether T is invertible or not. Justify your answer.

(c) For the adjoining graphic, use homogenous co-ordinates to find the new vertices after performing scaling about (7,3) with scale factors of ½ in the x - direction and 3 in the y - direction.

$R(T) = \text{span}\ (T(\beta)) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\}$

If T is one-to-one and onto then prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

(b) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear,

$T(1,1) = (1,-2)$

$T(-1, 1) = (2, 3)$.

What is $T(-1,5)$ and $T(x_1, x_2)$?

Find $[T]_\beta^\gamma$ if $\beta = \{(1,1), (-1,1)\}$ and $\gamma = \{(1,-2), (2,3)\}$.

(c) For the following linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

find bases for null space N(T) and range space R(T). Also, verify the dimension theorem.

(b) Consider the matrix :

$$A = \begin{pmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{pmatrix}$$

(i) Find the eigenvalue and the fundamental eigenvectors of A.

(ii) Is A diagonalizable? Justify your answer.

(c) Find the reduced row echelon form matrix B of the following matrix :

$$A = \begin{pmatrix} 1 & 2 & -2 & -11 \\ 2 & 4 & -1 & -10 \\ 3 & 6 & -4 & -25 \end{pmatrix}$$

and then give a sequence of row operations that convert B back to A.

3. (a) Let F_1 and F_2 be fields. Let $\mathcal{F}(F_1, F_2)$ denote the vector space of all functions from F_1 to F_2. A function $g \in \mathcal{F}(F_1, F_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in F_1$ and is called an odd

function if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

(b) Let W_1 and W_2 be subspaces of a vector space V.

(i) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(ii) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(c) (i) Let S_1 and S_2 are arbitrary subsets of a vector space V. Show that if $S_1 \subseteq S_2$ then span$(S_1) \subseteq$ span (S_2).

(ii) Let F be any field. Show that the vectors (1,1.0), (1,0,1) and (0,1,1) generate F^3.

4. (a) Define a linearly independent subset of a vector space V. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in$ span $(\{u_1, u_2, \ldots, u_k\})$ for some k, $(1 \leq k < n)$.

(b) Let V be a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of V. Prove that β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$, for unique scalars $a_1, a_2, \ldots, a_n$.

(c) Let F be any field. Consider the following subspaces of F^5:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \mid a_2 = a_3 = a_4 = 0, a_1 + a_5 = 0\}$$

Find bases and dimension for the subspaces W_1, W_2 and $W_1 \cap W_2$.

5. (a) Let V and W be vector spaces over a field F, and let $T : V \rightarrow W$ be a linear transformation. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then prove that