(d) Let f be defined on $\mathbb{R}$ and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function. (5)

5. (a) Let f be differentiable function on an open interval (a, b). Then show that f is increasing on (a, b) if $f'(x) \geq 0$. (5)

(b) If $y = e^{\tan^{-1}x}$, prove that (5)

$$(1 + x^2)y_{n+2} + (2(n + 1)x - 1)y_{n+1} + n(n + 1)y_n = 0.$$

(c) If $y = \cos(m \sin^{-1} x)$, find $y_n(0)$. (5)

(d) Stating Taylor's theorem find Taylor series expansion of e^x. (5)

6. (a) Find $\lim_{x \to +\infty} \left[x - \ln\left(x^2 + 1\right)\right]$. (5)

(b) Determine the position and nature of the double points on the curve (5)

$$x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0.$$

(c) Sketch a graph of the rational function showing the horizontal, vertical and oblique asymptote (if any) of $y = \dfrac{x^2 - 2}{x}$. (5)

(d) Sketch the curve in polar coordinates of $r = \sin 2\theta$. (5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1223 **F**

Unique Paper Code : 2352011202

Name of the Paper : CALCULUS

Name of the Course : **B.Sc. (H) Mathematics UGCF-2022**

Semester : II – DSC 5

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator is not allowed.

1. (a) State and prove the sequential criterion for the limit of a real valued function. (5)

(b) Use $\in - \delta$ definition of limit to establish the following limit : (5)

$$\lim_{x \to 2} \frac{1}{1 - x} = -1.$$

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as (5)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f has a limit only at $x = 0$.

(d) Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x \to c} f > 0$, then show that $f(x) > 0$ for all $x \in A \cap V_\delta(c)$, $x \neq c$. (5)

2. (a) If f is continuous at x_0 and g is continuous at $f(x_0)$ then prove that the composite function $g \circ f$ is continuous at x_0. (5)

(b) Let $f(x) = \frac{1}{x}\sin\frac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that f is discontinuous at 0. (5)

(c) State Intermediate Value Theorem. Prove that $xe^x = 1$ for some x in (0,1). (5)

(d) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$. (5)

3. (a) Prove that if a real valued function f is continuous on [a, b] then it is uniformly continuous on [a, b]. (5)

(b) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on (a, ∞) for $a > 0$ but it is not uniformly continuous on (0,1). (5)

(c) Let $f(x) = |x| + |x - 1|$, $x \in \mathbb{R}$. Draw the graph and give the set of points where it is not differentiable. Justify also. (5)

(d) Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$. (5)

4. (a) State and prove Mean Value Theorem. (5)

(b) State Intermediate Value Theorem for derivatives. Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.

(i) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0,2)$.

(ii) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0,2)$.

(5)

(c) Prove that $(\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. (5)