(c) Let W be the subspace of $\mathbb{R}^3$, whose vectors lie in the plane $2x - 5y + z = 0$. Find a basis for W and its orthogonal complement. (6)

27/5/23 (Eve)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 6095 E

Unique Paper Code : 32355202

Name of the Paper : Linear Algebra

Name of the Course : **Generic Elective – Mathematics [other than Maths (H)]**

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** questions by selecting any **two** parts from each question.

1. (a) If x and y are vectors in $\mathbb{R}^n$, then prove that

   (i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if and only if $x \cdot y = 0$.

(ii) If $(x + y) . (x - y) = 0$, then $||x|| = ||y||$.

(6½)

(b) If $x = [4, 0, -3]$ and $y = [3, 1, -7]$ be two vectors in $\mathbb{R}^3$, then decompose the vector y into two component vectors in directions parallel and orthogonal to the vector x. (6½)

(c) Define rank of a matrix. Also, find the rank of the matrix (6½)

$$\begin{bmatrix} 3 & 1 & 0 & 1 & -9 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

2. (a) Solve the following system of linear equations using Gaussian Elimination method

$$3x_1 - 6x_2 + 3x_4 = 9$$

$$-2x_1 + 4x_2 + 2x_3 - x_4 = -11$$

$$4x_1 - 8x_2 + 6x_3 + 7x_4 = -5$$

(6)

(c) Check whether the linear transformation $L : \mathcal{P}^3 \rightarrow \mathcal{P}^2$ defined by

$$L(a x^3 + b x^2 + c x + d) = a x^2 + b x + c$$

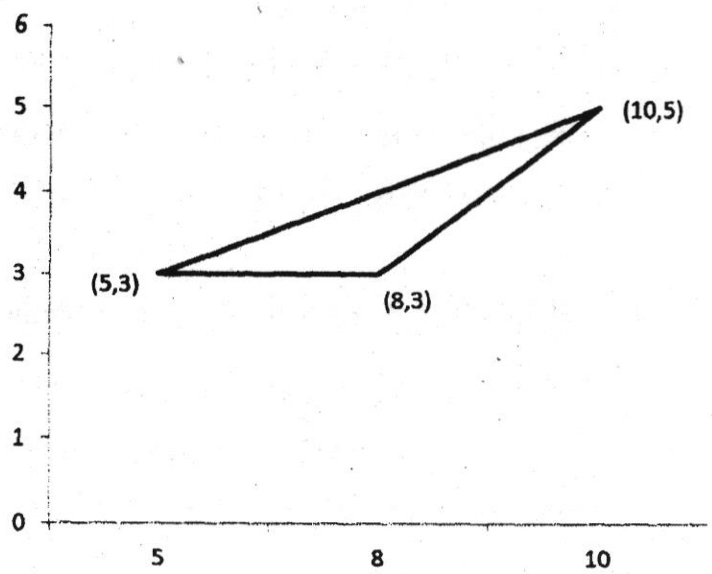
is an isomorphism or not. (6½)

6. (a) Find a least square solution for the following inconsistent system

$$\begin{pmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}.$$

(6)

(b) Let $W = \text{span}\{[8, -1, -4], [4, 4, 7]\}$ and $v = [1,2,3] \in \mathbb{R}^3$. Find $\text{proj}_W v$, and decompose v into $w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^{\perp}$. (6)



(b) State Dimension Theorem. Find a basis for Ker(L) and a basis for range(L) for the linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by (6½)

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \forall\, (x, y, z) \in \mathbb{R}^3$$

(b) Examine whether the matrix $A = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}$ is diagonalizable. (6)

(c) Let V be a vector space, and let W_1 and W_2 be subspaces of V. Show that

(i) $W_1 \cap W_2$ is also a subspace of V.

(ii) $W_1 + W_2$ defined by $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ is also a subspace of V. (6)

3. (a) Use the Simplified Span Method to find a simplified general form for all the vectors in span(S) where

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ is a subset of } \mathcal{M}_{22}.$$

Is the set S linearly independent? Justify. (6½)

(b) Define a basis for a vector space. Examine whether the set

$S = \{[3, 1, -1], [5, 2, -2], [2, 2, -1]\}$ forms a basis for $\mathbb{R}^3$? (6½)

(c) Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -3 \end{bmatrix}$. Using rank of A determine whether the homogeneous system $AX = 0$ has a non-trivial solution or not. If so, find the non-trivial solution. (6½)

4. (a) Let $S = \{[1, 0], [1, -3]\}$ and $T = \{[1, -1], [1, 1]\}$ be two ordered bases for $\mathbb{R}^2$. Find the transition matrix $P_{S \leftarrow T}$ from T-basis to S-basis. If v is in $\mathbb{R}^2$ and $[v]_T = [5,1]$, determine $[v]_S$. (6)

(b) Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator and $L([1, 0, 0]) = [-2, 1, 0]$, $L([0, 1, 0]) = [3, -2, 1]$ and $L([0, 0, 1]) = [0, -1, 3]$. Find $L([x,y,z])$ for any $[x,y,z] \in \mathbb{R}^3$. Also find $L([-3, 2, 4])$. (6)

(c) Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $L([x, y]) = [2x - y, x - 3y]$. Find the matrix for L with respect to the basis $T = \{[4, -1], [-7, 2]\}$ using the method of similarity. (6)

5. (a) For the graphic in the given figure, use ordinary coordinates in $\mathbb{R}^2$ to find new vertices after performing each indicated operation. Then sketch the figure that would result from this movement.

(i) Translation along the vector $[2, -1]$. (6½)

(ii) Reflection about the line $y = 2x$.