(d) Lehman-Scheffe's theorem (12½)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1154 A

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Name of the Paper : Statistical Inference

Name of the Course : **B.Sc. (H) Statistics**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

(Plus additional 30 minutes)

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **six** questions in all, selecting **three** questions from each **Section**.

SECTION A

1. (a) Define consistency of an estimator. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f.

$f(x,\theta) = \frac{1}{\theta} e^{\frac{-x}{\theta}};\ x > 0,\ \theta > 0$. Examine whether the following estimators are consistent for θ.

(i) $T_1 = \overline{X}$

(ii) $nX_{(1)}$

(b) Let X be distributed in Poisson form with parameter θ. Show that the only unbiased estimator of $e^{-(k+1)\theta}$ is $T = (-K)^X$ so that

$T(x) > 0$ if x is even

$T(x) < 0$ if x is odd.

Comment on the result. (6½+6)

2. (a) Define MVU estimator and efficiency of an estimator. Let T_0 be a minimum variance unbiased estimator and T_1 be an unbiased estimator of $\gamma(\theta)$ with efficiency e_θ. If ρ_θ is the correlation coefficient between T_0 and T_1. Show that

$\rho_\theta = \sqrt{e_\theta}$.

(b) State and prove Cramer-Rao inequality. Explain its significance. (6½+6)

9. (a) Discuss the method of construction of likelihood ratio test. Consider n Bemoull trials with probability of success p for each trial. Derive the likelihood ratio test for testing H_0: $p = p_0$ against H_1: $p > p_0$.

(b) Let X have a p.d.f. of the form:

$$f(x,\theta) = \frac{1}{\theta} e^{\frac{-x}{\theta}}; x > 0, \theta > 0$$

To test H_0: $\theta = 2$ against H_1: $\theta = 1$, use a random sample X_1 and X_2 of size 2 and define a critical region $C = \{(x_1, x_2): 9.5 \le x_1 + x_2\}$. Find (i) significance level of the test (ii) Probability of type II error and (iii) Power of the test.

(6½+6)

10. Describe any **three** of the following :

(a) Factorisation theorem

(b) Method of minimum Chi-square

(c) Optimum properties of ML estimators

8. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution, μ is known. Obtain UMPCR of size α for testing (i) $H_0: \sigma^2 = \sigma_0^2$ against $H_1: \sigma^2 > \sigma_0^2$ (ii) $H_0: \sigma^2 = \sigma_0^2$ against $H_1: \sigma^2 < \sigma_0^2$. Also show that there does not exist a UMPCR of size α for testing $H_0: \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$.

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a discrete distribution with p.m.f. f(x). According to H_0,

$$f(x) = \begin{cases} \dfrac{e^{-1}}{x!} \text{ if } x = 0,1,2,\ldots \\ 0 \quad \text{otherwise} \end{cases}$$

and according to H_1,

$$f(x) = \begin{cases} \dfrac{1}{2^{x+1}} \text{ if } x = 0,1,2,\ldots \\ 0 \quad \text{otherwise} \end{cases}$$

Obtain the critical region of the MP test of level α for H_0 against H_1. (6½+6)

3. (a) Describe the method of moments. For the double Poisson distribution

$$p(x) = P[X = x] = \frac{1}{2}\left[\frac{e^{-m_1} m_1^x}{x!} + \frac{e^{-m_2} m_2^x}{x!}\right]; x = 0,1,2,\ldots$$

Show that the estimates for m_1 and m_2 by the method of moments are

$m_1' \pm \sqrt{m_2' - m_1' - (m_1')^2}$ where m_1' and m_2' are the first two sample moments about origin of X.

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having p.d.f.

$f(x, \theta) = (\theta + 1)x^{\theta}$; $0 < x < 1$, $\theta > 0$. Find the ML estimator of θ. Also find sufficient estimator of θ. (6½+6)

4. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having p.d.f.

$f(x, \theta) = \frac{1}{\theta}; 0 < x < \theta, \theta > 0$. Show that the largest order statistic $X_{(n)}$ is complete sufficient statistic. Hence, find MVU estimator of θ.

(b) Define MVB estimator. Examine whether there exists an MVB estimator for the parameter θ based on a random sample of size n from a Cauchy distribution having

p.d.f. $f(x,\theta)=\dfrac{1}{\pi\left[1+(x-\theta)^2\right]};-\infty<x<\infty$. Also obtain the value of C-R lower bound. (6½+6)

5. (a) What do you mean by Blackwellisation process? State and prove Rao-Blackwell theorem.

(b) Illustrate with the help of an example :

(i) ML estimator may not always exist

(ii) ML estimator may not be unbiased

(6½+6)

SECTION B

6. (a) Let X be a Poisson distribution variate and the prior distribution of its parameter λ be Gamma distribution with p.d.f.

$$g(\lambda)=\frac{e^{-\alpha\lambda}\lambda^{\beta-1}\alpha^{\beta}}{\Gamma(\beta)};\lambda>0,\alpha>0,\beta>0.$$ Find the posterior distribution of λ. Also find the mean of the posterior distribution of λ.

(b) What are simple and composite hypotheses? Define most powerful test, uniformly most powerful test and unbiased critical region. State the theorem used to determine the most powerful critical region for testing a simple null hypothesis against a simple alternative hypothesis. (6½+6)

7. (a) Obtain 100(1 – α)% confidence limits (for large samples) based on a random sample of size n for the parameter λ of the Poisson distribution:

$$f(x,\lambda)=\frac{e^{-\lambda}\lambda^{x}}{x!};x=0,1,2,...$$

(b) Define pivotal quantity method. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having p.d.f. $f(x,\theta) = e^{-(x-\theta)};\ \theta \le x < \infty,\ -\infty < \theta < \infty$. Show that $P\left[X_{(1)}+\frac{1}{n}\log\alpha\le\theta\le X_{(1)}\right]=1-\alpha$.