Name of Course	:	CBCS(LOCF) B.Sc.(H)Mathematics
Unique Paper Code	:	32351301
Name of Paper	:	BMATH305-Theory of Real Functions
Semester	:	III
Duration	:	3 hours
Maximum Marks	:	75 Marks

Attempt any four questions. All questions carry equal marks.

1. Prove that $\lim_{x\to -1} \frac{x^2-5}{x^2+7} \neq \frac{-5}{7}$.

   Use $\varepsilon - \delta$ definition of limit to prove that $\lim_{x\to 1} \frac{x^3-3}{x^2+1} = -1$.

   Also prove that $\lim_{x\to 0} x^2 sgn(x)$ exists . Here sgn denotes the signum function.

2. Let $A \subseteq \mathbb{R}$, functions $f, g: A \longrightarrow \mathbb{R}$, c *be a cluster point of* A *and* $L \in \mathbb{R}$, $L \neq 0$.

   If $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = \infty$, then find $\lim_{x\to c} fg$.

   If $\lim_{x\to c} f = 0$ and $\lim_{x\to c} g = \infty$ then justify by an example that $\lim_{x\to c} fg$ need not be infinity.

3. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 , & if\ x\ is\ rational \\ 5x - 6 , & if\ x\ is\ irrational \end{cases}$$

   Find all the points at which f is continuous.

   Let $f: \mathbb{R} \to \mathbb{R}$ be an additive function ,that is, $f(a+b) = f(a) + f(b)$ for all $a, b \in \mathbb{R}$. Prove that if f is continuous at some point x_0 , then f is continuous at every point of $\mathbb{R}$.

4. Show that $f(x) = \frac{x-1}{x+1}$ is uniformly continuous on $[0, \infty)$ and $g(x) = Cos(\frac{1}{x})$ is not uniformly continuous on $(0, \infty)$.

   If f is continuous on $[0, 2]$ and $f(0) = f(2)$, then prove that there exists $x, y \in [0,2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.

5. Find the points of relative extrema of the following function on the specified domain $f(x) = |x^2 - 25|, \quad -7 \leq x \leq 7$.

   Prove that $ex \leq e^x$, for all $x \in \mathbb{R}$.

   Use Mean Value Theorem to find an approximate value of $\sqrt{51}$.

6. If $x \in [0,1]$ and $n \in \mathbb{N}$, show that

$$\left| ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

   Use this to approximate $\ln(1.5)$ with an error less than 0.01.

Use Taylor's theorem to prove that for all $x > 0$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} < e^x < 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} e^x .$$