S.N. Q.P. - 8171

Unique Paper Code	: **32351403**
Name of Course	: **CBCS (LOCF) B.Sc. (H) Mathematics**
Name of Paper	: **BMATH-410- Ring Theory and Linear Algebra-I**
Semester	: **IV**
Duration	: **3:30 hours**
Maximum Marks	: **75 Marks**

(Write your Roll No. on the top immediately on receipt of this question paper)

*Attempt any **two** parts from each question. All questions are compulsory.*

1. (a) A nonempty subset S of a ring R is a subring of R if and only if
   (i) $a, b \in S$ implies $a - b \in S$ and
   (ii) $a, b \in S$ implies $ab \in S$. $(5\frac{1}{2})$

   (b) Define a unit in a ring. Determine all units of $\mathbb{Z}[i]$ and $\mathbb{Z}[x]$. $(5\frac{1}{2})$

   (c) Show that every finite integral domain is a field. Give an example of an infinite integral domain which is not a field. $(6\frac{1}{2})$

   (d) Define characteristic of a ring. Show that the characteristic of an integral domain is either zero or a prime number. $(5\frac{1}{2})$

2. (a) If m and n are positive integers, $k = lcm(m, n)$. Show that $m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$. (6)

   (b) Prove that the intersection of any collection of subrings of a ring R, is a subring of R. Is the union of collection of subrings of a ring, a subring? Justify. (6)

   (c) Show that every nonzero element of $\mathbb{Z}_n$ is a unit or a zero divisor. (6)

   (d) Show that the set $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is an integral domain. (6)

3. (a) Let R be a ring and let A be a subring of R. Show that the set of cosets $\{r + A | r \in R\}$ is a ring under the operations $(s + A) + (t + A) = s + t + A$ and $(s + A)(t + A) = st + A$ if and only if A is an ideal of R. $(6\frac{1}{2})$

   (b) If n is an integer greater than 1, show that $< n >= n\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ if and only if n is prime. $(6\frac{1}{2})$

   (c) Let R be a commutative ring with unity and let A be an ideal of R. Then show that R/A is an integral domain if and only if A is prime ideal. $(6\frac{1}{2})$

   (d) Show that $\frac{\mathbb{Z}_3}{<x^2+x+1>}$ is not a field. $(6\frac{1}{2})$

4. (a) Let ϕ be a ring homomorphism from a ring R to a ring S. Let A be a subring of R and let B be an ideal of S. Then show that
   (i) ϕ is an isomorphism if and only if ϕ is onto and $\ker(\phi) = \{r \in R | \phi(r) = 0\} = \{0\}$.
   (ii) $\phi^{-1}(B) = \{r \in R \mid \phi(r) \in B\}$ is an ideal of R. (3+3)

(b) Determine all ring homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{20}$. (6)

(c) Let n be an integer with decimal representation $a_k a_{k-1} \dots a_1 a_0$. Prove that n is divisible by 9 if and only if $a_k + a_{k-1} + \cdots + a_1 + a_0$ is divisible by 9. (6)

(d) Let R be commutative ring of prime characteristic p. Show that the Frobenius map $x \to x^p$ is a ring homomorphism from R to R. (6)

5. (a) Let $v_1 = (1,-1,2), v_2 = (1,-2,1)$ and $v_3 = (1,1,4)$ be vectors in $\mathbb{R}^3$. Show that $span(\{v_1, v_2, v_3\}) = span(\{v_1, v_2\})$. (6)

(b) Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. ~~Then~~ Prove $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$. (6)

(c) Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Find a basis for W. (6)

(d) Let W_1 and W_2 be subspaces of a vector space V. Prove that V is direct sum of W_1 and W_2 if and only if each vector in V can be uniquely expressed as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$. (6)

6. (a) Let V and W be vectors spaces, and let $T: V \to W$ be linear transformation. Suppose that $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W. $(6\frac{1}{2})$

(b) Let V, W and Z be finite-dimensional vector spaces with ordered bases α, β and γ respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations. Then $(6\frac{1}{2})$

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

(c) Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U: P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b),$$

where $g(x) = 3 + x$. Let β and γ be standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively. Compute $[UT]_\beta^\gamma$. $(6\frac{1}{2})$

(d) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be linear defined by

$$T(a, b, c) = (3a - 2c, b, 3a + 4b).$$

Determine whether T is invertible. Justify your answer. If so, find T^{-1}. $(6\frac{1}{2})$