(d) Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < R_1 < R$, show that the power series converges uniformly on $[-R_1, R_1]$. Also, show that the sum function f(x) is continuous on the interval $(-R, R)$. (6.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1377 **A**

Unique Paper Code : 32351402

Name of the Paper : BMATH-409; Riemann Integration and Series of Functions

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 hours + 30 minutes Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Let f be a bounded function on [a, b]. Define integrability of *f* on [a, b] in the sense of Riemann. (6)

(b) Prove that every continuous function on [a, b] is integrable. Discuss about the integrability of discontinuous functions. (6)

(c) Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on [–1, 1], Show that $\left|\int_{-1}^{1} f(t)dt\right| \leq 2$. (6)

(d) Let f(x) = x for rational x; and f(x) = 0 for irrational x. Calculate the upper and lower Darboux integrals of f on the interval [0, b]. Is f integrable on [0, b]? (6)

2. (a) State Fundamental Theorem of Calculus II. Use it to calculate

$$\lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt. \quad (6.5)$$

(b) Let f be defined as (6.5)

$$f(t)=\begin{cases}0, & t<0\\ t, & 0\leq t\leq 1\\ 4, & t>1\end{cases}$$

(i) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}(x+1)^n$

(ii) $\sum_{n=0}^{\infty} x^{n!}$ (6.5)

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Show that the function f is differentiable on (–R, R) and

$$f'(x)=\sum_{n=1}^{\infty} na_n x^{n-1} \text{ for } |x| < R. \quad (6.5)$$

(c) Show that

(i) $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+----$ for $|x|<1$

(ii) $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}------.$ (6.5)

5. (a) (i) State and prove the Cauchy Criteria for uniform convergence of series. (3.5)

(ii) Show that the series $\sum_0^\infty (1-x)x^n$ is not uniformly convergent on [0, 1] (3)

(b) Show that $\sum \frac{(-1)^n}{n^p} \frac{x^{2n}}{\left(1+x^{2n}\right)}$ converges absolutely and uniformly for all values of x if $p > 1$.

(c) Is the sequence $< f_n >$ where $f_n = \frac{\sin(nx+n)}{n}$, uniformly convergent on $\mathbb{R}$? Justify. (6.5)

(d) If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in N$ and $\sum f_n$ converges to f uniformly on D then prove that f is continuous on D. (6.5)

6. (a) Find the radius of convergence and exact interval of convergence of the following power series :

(i) Determine the function $F(x) = \int_0^x f(t)dt$.

(ii) Sketch F. Where is F continuous?

(iii) Where is F differentiable? Calculate F′ at points of differentiability.

(c) State and prove Intermediate value theorem for Integral Calculus. Give an example to show that condition of continuity of the function cannot be relaxed. (6.5)

(d) Let f be a continuous function on $\mathbb{R}$. Define

$$G(x) = \int_0^{\sin x} f(t)dt \text{ for } x \in \mathbb{R}.$$

Show that G is differentiable on $\mathbb{R}$ and compute G′. (6.5)

3. (a) Let $\beta(p, q)$ (where $p, q > 0$) denotes the beta function, show that

$$\beta(p,q) = \int_{0^+}^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} du = \int_{0^+}^{1} \frac{v^{p-1} + v^{q-1}}{(1+v)^{p+q}} dv. \quad (6)$$

(b) Determine the convergence and divergence of the following improper integrals

(i) $\int_0^1 \frac{dx}{x(\ln x)^2}$

(ii) $\int_{-1}^{\infty} \frac{dx}{x^2+4x+6}$ (6)

(c) Define Improper Integral of type II.

Show that the improper integral $\int_1^{\infty} \frac{dx}{x^p}$ converges iff $p > 1$. (6)

(d) Show that the improper integral $\int_{\pi}^{\infty} \frac{\text{Sin } x}{x} dx$ is convergent but doesn't converge absolutely. (6)

4. (a) Let $< f_n >$ be a sequence of integrable functions on $[a,b]$ and suppose that $< f_n >$ converges uniformly on $[a, b]$ to f. Show that is f is integrable. (6)

(b) Define

(i) pointwise convergence of sequence of functions

(ii) uniform convergence of a sequence of functions

(iii) If $A \subseteq \mathbb{R}$ and $\varnothing: A \to \mathbb{R}$ then define uniform norm of $\varnothing$ on A. (6)

(c) (i) Discuss the pointwise and uniform convergence of $f_n(x) = \frac{x}{n}$ for $x \in \mathbb{R}$, $n \in N$.

(ii) Show that the sequence $< f_n >$ where $f_n(x) = \frac{n}{x+n}$, $x \geq 0$ is uniformly convergent in any finite interval. (6)

(d) (i) Show that the sequence $< f_n >$ where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ uniformly convergent on $[0, \pi]$.

(ii) Discuss the pointwise and uniform convergence of the sequence $g_n(x) = x^n$ for $x \in \mathbb{R}$, $n \in N$. (6)