Name of Course : B.Sc. (H) Mathematics CBCS (LOCF)

Unique Paper Code : 32351303

Name of Paper : BMATH307 - Multivariate Calculus

Semester : III

Duration : 3 hours

Maximum Marks : 75 Marks

Attempt any four questions. All questions carry equal marks.

1. Let $f(x,y)=\begin{cases} \dfrac{xy+y^3}{x^2+y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0). \end{cases}$

   (a) Show that $f_x(0,0)$ and $f_y(0,0)$ both exist but f is not differentiable at $(0,0)$.

   (b) Find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q\left(\frac{1}{2},2\right)$.

   (c) Find the direction in which f decreases most rapidly and increases most rapidly at the point $(2,1)$.

   (d) Find the tangent plane to the given surface $z=f(x,y)$ at the point $(0,2,2)$.

2. Let $f(x,y)=x^2-4xy+y^3+4y$.

   (a) Find the absolute extrema of $f(x,y)$ on the closed bounded rectangular region $0\leq x\leq 3,\ 0\leq y\leq 2$

   (b) Use the method of constrained optimization to find the largest and the smallest value of $f(x,y)$ subject to the constraint $x+y=2$.

   (c) Compare the results obtained in parts (a) and (b) and interpret it.

3. (a) Let R be the region which lies between two squares of sides 2 and 4 with center at the origin and sides parallel to the coordinate axes. Compute the double integral $\iint_R e^{x+y}\,dA$.

(b) Evaluate

$$\iint_D \sin^2(x+y)\,dydx$$

where D is the parallelogram with vertices $(0,\pi)$, $(\pi,0)$, $(\pi,2\pi)$, $(2\pi,\pi)$.

(c) Compute $\iiint_D \frac{z}{(x-1)^2+(y-1)^2}\,dV$, where D is the solid bounded by the surface $(x-1)^2+(y-1)^2=2z$ and the plane $z=2$.

4. Let the vector field $\vec{F}$ be given as

$$\vec{F}(x,y,z)=\left(y^2\cos x+z^3\right)\hat{i}+(2y\sin x-4)\,\hat{j}+\left(3xz^2+2\right)\hat{k}$$

(a) Show that $\vec{F}$ is conservative.

(b) Find the scalar potential for $\vec{F}$.

(c) What will be the work done in moving an object in this force field from the point $P(0,1,-1)$ to the point $Q\left(\frac{\pi}{2},-1,2\right)$.

(d) If the path P to Q is traversed through ten additional in between points then what will be the effect on the work done. Give reason for your answer.

(e) Evaluate $\int_C \vec{F}\cdot d\vec{R}$, where C is the line segment $x=1$, $y=1$, $0\le z\le 1$.

5. Let D be a ball of radius 3 with center at the origin and $\vec{F}$ be the vector field given by

$$\vec{F}(x,y,z)=2xy\,\hat{i}+yz^2\,\hat{j}+xz\,\hat{k}.$$

(a) Use spherical coordinates to compute $\iiint_D \nabla \cdot \vec{F}\, dV$.

(b) If S denotes the upper half of the ball D, then verify Stokes' Theorem for $\vec{F}$.

6. (a) Let the force field $\vec{F}$ be given by

$$\vec{F}(x, y, z) = xyz\left(\hat{i} + \hat{j} + \hat{k}\right).$$

If $\vec{N}$ is the outward drawn normal, then use Divergence theorem to evaluate $\iint_S \vec{F} \cdot \vec{N} dS$,

where S is the surface of the box: $0 \le x \le 2,\ 0 \le y \le 1,\ 0 \le z \le 4$.

(b) Evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} dzdydx$$

using

(i) Cylindrical coordinates

(ii) Spherical polar coordinates.