Name of Course	: **CBCS(LOCF) B.Sc. (H) Mathematics**
Unique Paper Code	: **32351501**
Name of Paper	: **BMATH511-Metric Spaces**
Semester	: **V**
Duration	: **3 hours**
Maximum Marks	: **75 Marks**

Attempt any four questions. All questions carry equal marks.

1. Let $X = C[0,2]$, the space of all continuous functions defined on $[0,2]$. Let

$$d_1(f,g) = \int_0^2 |f(x) - g(x)|dx \text{ and } d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in [0,2]\}.$$

Compute the distance $d_1(f,g)$ and $d_\infty(f,g)$ where

$$f(x) = \begin{cases} \sin x : 0 \le x < \frac{\pi}{4} \\ \frac{1}{\sqrt{2}} : \frac{\pi}{4} \le x \le 2 \end{cases} \text{ and } \quad g(x) = \begin{cases} \cos x : 0 \le x < \frac{\pi}{4} \\ \frac{1}{\sqrt{2}} : \frac{\pi}{4} \le x \le 2 \end{cases}.$$

Let X be any non-empty subset of $\mathbb{R}$. Define a function $d : X \times X \to [0,\infty)$ as

$$d(x,y) = \begin{cases} |x-y|, & if\ |x-y| \le 1 \\ 1, & if\ |x-y| \ge 1 \end{cases}.$$

Show that d is a metric on X and d is bounded.

Let $X = \mathbb{R}^3$ and d be the metric on $\mathbb{R}^3$ given by $d(x,y) = \sum_{i=1}^{3} |x_i - y_i|$ where $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$. Let $\{x^{(n)}\}$ be a sequence in $\mathbb{R}^3$ where $x^{(n)} = \left(\frac{n}{n+1}, \frac{1}{n^3}, 1 - \frac{1}{n}\right), n \in \mathbb{N}$. Is $\{x^{(n)}\}$ convergent in $\mathbb{R}^3$? If yes, find the limit.

Is $d(x,y) = |x-y|^3$ a metric on $\mathbb{R}$? Justify your answer.

(6+6.75+4+2)

2. Let $a, b \in \mathbb{R}$ and $a < b$. Show that the open interval (a,b) is an incomplete subspace of $\mathbb{R}$. Let Y be a finite subset of $\mathbb{R}$ with usual metric. Is Y open in $\mathbb{R}$? Justify. If not, then give an example of a metric space in which a finite set may be open.

Let $(C[0,1], d_1)$ be a metric space with the metric defined by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx.$$

Let $\{h_n\}$ be a sequence in $C[0,1]$ defined by

$$h_n(x) = x^{1/n}, \qquad x \in [0,1]$$

Show that $h_n \to h$ in $(C[0,1], d_1)$, where $h(x) = 1$ for all $x \in [0,1]$. Does the same statement hold in $(C[0,1], d_\infty)$ where $d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$? Justify.

Give an example of a set which is neither open nor bounded. Justify your answer.

(9+6+3.75)

3. Find the closure and interior of the set $A = \{(x,y): xy = 1\}$ as a subset of $\mathbb{R}^2$ (equipped with the Euclidean metric).

If D is an open subset of $\mathbb{R}$, which contains all rational numbers lying between 0 and 2, then does $\sqrt{2} \in D$? Justify your answer.

Consider the set $X = A_1 \cup A_2$, where $A_1 = (0,1)$ and $A_2 = [2,3)$. Show that A_1 and A_2 are both open as well as closed in X.

Let (X,d) be a metric space and $B = S(x_0, r)$ be the open ball with centre at $x_0 \in X$ and radius $r > 0$. Show that $d(B) \le 2r$, where d(B) denotes the diameter of the set B. Give an example to show that in general the equality may not hold in $d(B) \le 2r$.

(4+4+4+6.75)

4. Let d denote the Euclidean metric in $\mathbb{R}^2$ and d_1 be the metric defined in $\mathbb{R}^2$ by

$$d_1(\mathrm{x},\mathrm{y}) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for} \quad \mathrm{x} = (x_1, x_2), \mathrm{y} = (y_1, y_2) \in \mathbb{R}^2 .$$

Let $f: (\mathbb{R}^2, d) \to (\mathbb{R}^2, d_1)$ be given by $f(x_1, x_2) = (2x_1, x_2)$. Prove that f is continuous on $\mathbb{R}^2$.

Let $\mathbb{R}$ be equipped with the usual metric and $g: \mathbb{R} \to \mathbb{R}$ be continuous. Let $A = \{x \in \mathbb{R}: g(x) \ge 0\}$. Show that A is closed in $\mathbb{R}$. Is A complete with respect to the induced metric?

For the subset $E = \left\{\left((-1)^n \frac{1}{n}, (-1)^n\right) : n \in \mathbb{N}\right\}$ of $\mathbb{R}^2$ (equipped with the Euclidean metric), find $\bar{E}$.

Let (X,d) and (Y,ρ) be two metric spaces and $h: X \to Y$ be a bijection. Show that h is a homeomorphism if and only if for all subsets A of X,

$$h(\bar{A}) = \overline{h(A)} .$$

(6+3.75+3+6)

5. Let $X = (-1, 1)$ and $Y = (0, 1)$ be subsets of the usual metric space $\mathbb{R}$. Are the spaces X and Y homeomorphic to each other? Justify your answer. What if we take $X = \mathbb{R}$ and $Y = (0, 1)$? Justify your answer.

Show that isometry from (X, d_X) into (Y, d_Y) is injective. Is every isometry from (X, d_X) onto (Y, d_Y) is a homeomorphism? Justify.

Prove that every contraction map on a metric space is uniformly continuous.

Let $\mathbb{R}^2$ be the Euclidean metric space. If $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a map defined by $T(x, y) = (x, 0)$. Prove or disprove that T is a contraction map. Also, find the fixed point(s) of T.

(6+4+3.75+5)

6. Let A and B be two connected subsets of X and $A \cap B \neq \phi$. Show that $A \cup B$ is connected. In $\mathbb{R}^2(eqquiped\ with\ the\ Euclidean\ metric)$, find if the union of $A = \{(x, y): x^2 + y^2 \leq 1\}$ and $B = \{(x, y): (x-2)^2 + y^2 < 1\}$ is connected or not? Justify.

Let (X, d) be a compact metric space and let d_1 be the metric on X defined by

$$d_1\,(x, y) = \min\{1, d(x, y)\}, \qquad x, y \in X$$

Then prove that (X, d_1) is compact.

Prove that every continuous real valued function f on a compact metric space attains its infimum. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$. Find the infimum of f. Is the above statement applicable for $f(x) = \frac{1}{1+x^2}$? Justify.

Examine the compactness of the set $B = \{(x, y) \in \mathbb{R}^2: x = 0\}$.

(6+4+6+2.75)