Name of the course : **CBCS B.Sc. (H) Mathematics**

Unique Paper Code : **32351302**

Name of Paper : **BMATH306-Group Theory-1**

Semester : **III**

Duration : **3 hours**

Maximum Marks : **75 Marks**

Attempt any four questions. All questions carry equal marks.

1. Show that the set S of all ordered pairs (a, b) of non-zero real numbers is an abelian group under the multiplication defined by

$$(a, b)(c, d) = (ac, bd) \ \forall \ a, b, c, d \in S$$

   Consider the group $G = GL(2, \boldsymbol{R})$ under multiplication. Then find the centralizer of $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Also, find the center of G.

   Let $A = \begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix}$. Find A^{-1} in $SL(2, Z_5)$. Verify the answer by direct calculation.

2. Find all the subgroups of $\boldsymbol{Z}$:
   a) containing $20\boldsymbol{Z}$.
   b) contained in $20\boldsymbol{Z}$.

   Prove that an abelian group which contains two distinct elements which are their own inverses must have a subgroup of order 4.

   Suppose a group contains elements a and b such that $|a| = 4$ and $|b| = 5$ and that $a^3 b = ba$. Find $|ab|$.

3. State Cayley's theorem and verify theorem for $U(10)$.

   Let a and b be elements of a group G. If $O(a) = 12$, $O(b) = 22$ and $< a > \cap < b > \neq \{e\}$. Prove that $a^6 = b^{11}$.

   Find a non-cyclic group of order 4 in $U(40)$.

4. Let p be a prime. If a group has more than $(p-1)$ elements of order p. Then prove that the group cannot be cyclic.

   Let $\beta = (1\ 2\ 3)(1\ 4\ 5)$. Write β^{99} as a cycle.

   Given a permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$

   a) Write α as product of disjoint cycle.
   b) Find $|\alpha|$.
   c) Find α^{-1} and verify by calculation.

5. Let G be the additive group $\boldsymbol{R} \times \boldsymbol{R}$ and $H = \{(x, x) : x \in \boldsymbol{R}\}$ be a subgroup of G. Give a geometric description of cosets of H.

   If N is a normal subgroup of order 2 of a group G then show that $N \subseteq Z(G)$.

   If H is a subgroup of a group G such that $(aH)(Hb)$ for any $a, b \in G$ is either a left or a right coset of H in G, prove that H is normal.

6. If $\emptyset$ be a homomorphism from Z_{30} onto a group of order 5, determine $Ker\ \emptyset$.

   Let N be a normal subgroup of a group G. If N is cyclic subgroup of G then prove that every subgroup of N is normal in G.

   Prove that the mapping from $x \to x^6$ from $\boldsymbol{C}^*$ to $\boldsymbol{C}^*$ where $\boldsymbol{C}^*$ denotes the set of non -zero complex numbers is a homomorphism. What is the kernel?