prove that it is the Taylor series for the function f(z) in power of $z - z_0$. (6.5)

6. (a) For the given function $f(z) = \frac{z+1}{z^2+9}$ find the poles, order of poles and their corresponding residue. (6)

(b) Write the two Laurent Series in powers of z that represent the function $f(z) = \frac{1}{z+z^3}$ in certain domains and specify those domains. (6)

(c) Suppose that $z_n = x_n + iy_n$, (n = 1,2,3,...) and $S = X + iY$. Then prove that

$\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. (6)

(d) Define residue at infinity for a function f(z). If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then prove that

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \quad (6)$$

[This question paper contains 6 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1114 **A**

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. 3. **All** questions are compulsory.
3. Attempt **two** parts from each question.

1. (a) Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi$ is mapped by the transformation $w = z^2$ and $w = z^3$. (6)

(b) (i) Find the limit of the function $f(z) = \frac{(\bar{z})^2}{z}$ as z tends to 0.

(ii) Show that $\lim_{z\to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i} = 2\sqrt{3}i.$ (3+3=6)

(c) Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \bar{z}^2/z & when\ \ z \neq 0 \\ 0 & when\ \ z = 0 \end{cases}$$

Verify that the Cauchy-Riemann equations are satisfied at the origin z = (0,0). (6)

(d) If $\lim_{z\to z_0} f(z) = F$ and $\lim_{z\to z_0} g(z) = G$, prove that

$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{F}{G}$ if $G \neq 0$. (6)

2. (a) Find the values of z such that

(i) $e^z = 1 + \sqrt{3}i$, (ii) $e^{(2z-1)} = 1$. (3.5+3=6.5)

(b) Show that the roots of the equation $cosz = 2$ are $z = 2n\pi + icosh^{-1}2$ $(n = 0, \pm 1, \pm 2, \ldots)$, Then express them in the form $z = 2n\pi \pm i\ln(2+\sqrt{3})$ $(n = 0, \pm 1, \pm 2, \ldots)$. (3.5+3=6.5)

(c) Show that (3.5+3=6.5)

(i) $\log(1+i)^2 = 2Log(1+i)$,

(d) (i) State Liouville's theorem. Is the function f(z) = cos z bounded? Justify.

(ii) Is it true that 'If p(z) is a polynomial in z then the function f(z) = 1/p(z) can never be an entire function'? Justify (4.5+2=6.5)

5. (a) If a series $\sum_{n=1}^{\infty} z_n$ of complex numbers converges then prove $\lim_{n\to\infty} z_n = 0$. Is the converse true? Justify. (6.5)

(b) Find the integral of $\int_C \frac{\cosh \pi z}{z^3+z}$ where C is the positively oriented circle |z| = 2. (6.5)

(c) Find the Taylor series representation for the function $f(z) = \frac{1}{z}$ about the point $z_0 = 2$. Hence prove that $\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$ for $|z-2| < 2$. (6.5)

(d) If a series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to f(z) at all points interior to some circle $|z - z_0| = R$, then

(ii) $\log(-1+\sqrt{3}i) = ln2 + 2\left(n+\frac{1}{3}\right)\pi i \ (n = 0, \pm 1, \pm 2, \dots)$

(d) Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \ (n = 0, \pm 1, \pm 2, \dots)$. (6.5)

3. (a) (i) State mean value theorem of integrals.

Does it hold true for complex valued functions? Justify.

(ii) Evaluate $\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta$. (3+3=6)

(b) Parametrize the curves C_1 and C_2, where

C_1: Semicircular path from −1 to 1

C_2: Polygonal path from the vertices −1, −1+i, 1+i and 1

Evaluate $\int_{C_1} z\,dz$ and $\int_{C_2} z\,dz$. (3+3=6)

(c) For an arbitrary smooth curve C: $z = z(t)$, $a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integrals

(i) $\int_{z_1}^{z_2} z\,dz$ and

(ii) $\int_{z_1}^{z_2} dz$

depend only on the end points of C. (3+3=6)

(d) State ML inequality theorem. Use it to prove that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$, where C denotes the line segment from z = i to z = 1. (2+4=6)

4. (a) A function f(z) is continuous on a domain D such that all the integrals of f(z) around closed contours lying entirely in D have the value zero. Prove that f(z) has an antiderivative throughout D. (6.5)

(b) State Cauchy Goursat theorem. Use it to evaluate the integrals

(i) $\int_C \frac{1}{z^2+2z+2} dz$, where C is the unit circle $|z| = 1$

(ii) $\int_C \frac{2z}{z^2+2} dz$, where C is the circle $|z| = 2$

(2.5+2+2=6.5)

(c) State and prove Cauchy Integral Formula.

(2+4.5=6.5)