Name of Course	:	**CBCS (LOCF) B.Sc. (H) Mathematics**
Unique Paper Code	:	**32351301**
Name of Paper	:	**BMATH305 – Theory of Real Functions**
Semester	:	**III**
Duration	:	**3 hours**
Maximum Marks	:	**75 Marks**

Attempt any four questions. All questions carry equal marks.

1. Use the ϵ-δ definition of the limit to show that

$$\lim_{x\to 1}\frac{x^3-2x+4}{x^2+4x-3}=\frac{3}{2}.$$

Use the sequential criteria for limits, to show that the following limit does not exist

$$\lim_{x\to 2}\operatorname{sgn}\left(\cos\left(\frac{1}{(x-2)^3}\right)\right),$$

where sgn is the signum function.

2. Let $r, L\in\mathbb{R}$ and $f:(-\infty,r)\to\mathbb{R}$ be a function. Prove that $\lim_{x\to-\infty} f(x)=L$ if and only if, for every sequence $<x_n>$ in $(-\infty,r)$ such that if $\lim_{n\to\infty} x_n=-\infty$, then the sequence $\langle f(x_n)\rangle$ converges to L.

3. Suppose $f:[a,b]\to\mathbb{R}$ is a monotonically increasing function. If f assumes every value between $f(a)$ and $f(b)$ at least once, then show that f is continuous on $[a, b]$.

Suppose $f:\mathbb{R}\to\mathbb{R}$ is a continuous function on $\mathbb{R}$. Let $a,b\in\mathbb{R}$ be such that $a<b$, then prove that $f^{-1}(\,]a,b[\,)$ is an open interval in $\mathbb{R}$, where $]a, b[$ denotes an open interval in $\mathbb{R}$.

4. Prove that the function $\sin(x^2)$ is not uniformly continuous on $[0,\infty)$. However, it is uniformly continuous on $[0,a]$, where $a>0$ is any fixed real number.

Suppose $f:[0,2\pi]\to\mathbb{R}$ is continuous and $f(0)=f(2\pi)$. Prove that there exists at least one point $c\in[0,\pi]$ such that $f(c)=f(c+\pi)$.

5. Let $f(x)=|\cos x|, x\in(0,2\pi)$. Determine that where the function f is differentiable, and where it is not differentiable in the interval $(0,2\pi)$. Also, find the derivative at the points of differentiability.

Let f be a function defined on the real line $\mathbb{R}$ and suppose that it satisfies the condition

$$|f(x)-f(y)|\le(x-y)^2 \qquad \text{for all } x,y\in\mathbb{R}.$$

Prove that f is a constant function.

6. Using the Taylor's theorem, find the approximate value of $\sin(0.4)$ with the error value being calculated up to less than 10^{-3}.

Let $x \in [0,1]$ and $n \in \mathbb{N}$, show that the following inequality holds

$$\left| e^x - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!}\right)\right| < \frac{3}{(n+1)!}.$$

Using this inequality, approximate the value of $\sqrt{e}$ with an error value determined to be less than 10^{-2}.