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This question paper contains 7 printed pages]

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S. No. of Question Paper : **7506**

Unique Paper Code : **32371501** **J**

Name of the Paper : **Stochastic Processes and Queueing Theory**

Name of the Course : **B.Sc. (Hons.) Statistics**

Semester : **V**

Duration : 3 Hours Maximum Marks : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *five* questions in all.

Section **I** is compulsory.

Attempt *four* more questions, selecting *two* questions from each of Sections **II** and **III**.

Use of simple calculator is allowed.

Section I

1. Attempt any *five* parts :

   (*a*) If X has a zero truncated Poisson distribution with zero class missing then obtain the probability generating function of X.

(*b*) Let X be a non-negative integral valued random variable with $P(X = k) = p_k$; $k = 0, 1, 2 ....$, and probability generating function $P(s)$. Find the generating function of $P(X > k + 1)$.

(*c*) Consider the process $X(t) = \sum_{r=1}^{k} (A_r \cos\theta_r t + B_r \sin\theta_r t)$ where A_r, B_r are uncorrelated random variables with mean 0 and variance σ^2 and θ_r are constants. Is the process $\{X(t), t \in T\}$ covariance stationary ?

(*d*) In a series of Bernoulli trials with probability of success p, let X denote the number of failures preceding the first success and Y, the number of failures following the first success and preceding the second success, then show that the p.g.f. of $Z = X + Y$ is $P(s,s) = \left(\frac{p}{1-qs}\right)^2$. Hence, obtain an expression for $P(Z = r)$.

7. (*a*) An airline organization has one reservation clerk on duty in its local branch. The clerk handles information regarding passenger reservations and flight timings. Assume that the number of customers arriving during any given period is in a Poisson fashion with an arrival rate of 8 per hour and that the reservation clerk can serve a customer in 6 minutes on an average with an exponentially distributed service time. Calculate the following :

(*i*) Probability that the system is busy,

(*ii*) Average time that a customer spends in the system,

(*iii*) Average queue length.

(*b*) For the model (M/M/1) : (N/FIFO), derive the steady state equations and find the probability distribution of number of customers in the system. Hence obtain the expected queue length. 7,8

Section III

5. (*a*) The births in hospital X in Agra occur in accordance with Poisson process with parameter λ. The probability that an individual born is male is p and the probability of a female child born is q. Prove that the male births form a Poisson process with parameter λp and the female births form a Poisson process with parameter λq.

(*b*) Derive the differential-difference equation for the linear growth process with immigration having $\lambda_n = n\lambda + a$, $\mu_n = n\mu$, when process starts with only i individuals at time $t = 0$. Show that $M(t) = E[X(t)]$ satisfies the differential equation $M'(t) = (\lambda - \mu)\, M(t) + a$. 7,8

6. (*a*) Prove that the random variable defined as the number of arrivals to a system in time t has a Poisson distribution with a mean arrival rate. Also state the underlying assumptions.

(*b*) Describe the classical ruin problem. Obtain the expected duration of the game. 7,8

(*e*) Let us suppose that the passengers arrive (singly) at a bus stand in accordance with a Poisson Process with parameter λ_1 and the buses arrive in accordance with a Poisson Process with parameter λ_2 then obtain the distribution of excess of passengers over buses in an interval t.

(*f*) In the classical ruin problem show that a fair game always remains fair and no unfair game can be changed to fair game.

(*g*) Show that the initial distribution together with transition matrix completely defines a Markov chain. 5×3

Section II

2. (*a*) Let a_n be the probability that a sequence of n Bernoulli trials result in an even number of successes. Find the generating function $A(s)$ of $\{a_n\}$. Hence, show that $a_n = \frac{1}{2}\{1 + (q - p)^n\}$, where p is probability of success in each trial.

(*b*) Let X_i, $i = 1, 2, 3, \ldots$ be identically and independently distributed random variables with $P(X_i = k) = p_k$ and p.g.f. $P(s) = \sum_{k=0}^{\infty} p_k s^k$. Let $S_N = X_1 + X_2 + \ldots + X_N$ where N is a random variable independent of the X_i's. Let the distribution of N be $P(N = n) = g_n$ and the probability generating function of N be $G(s) = \sum_{n=0}^{\infty} g_n s^n$. Obtain the probability generating function of S_N. Hence, obtain $E(S_N)$ and Variance (S_N). 7,8

3. (*a*) Show that in an irreducible Markov chain, all the states are of the same type.

(*b*) Consider the Markov chain with transition probability matrix with state space {1, 2, 3, 4, 5}.

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Is the chain irreducible ? Classify the states of the chain. 7,8

4. (*a*) Let $\{X_n, n \geq 0\}$ be a Markov chain with $S = \{0, 1, 2\}$, transition probability matrix :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \end{array}$$

and the initial distribution $P[X_0 = i] = \frac{1}{3}$, $i = 0, 1, 2$.

Find the probabilities :

(*i*) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$

(*ii*) $P(X_1 = 1)$

(*iii*) $P(X_2 = 1, X_0 = 0)$

(*b*) A man can either drive a car or catch a train to go to his office every day. He never goes two days in a row in train, but if he drives one day, then the next day he is equally likely to travel by train as he is to drive again. On the first day of the week he throws a die and drives to his office if and only if "6" appears. Find the probability that :

(*i*) he goes by train to his office on the third day of the week.

(*ii*) he catches a train in the long run. 7,8