(*c*) Find the cumulative distribution function for the following pdf :

$$f(x) = \begin{cases} 1/3 & 0 < x < 1 \\ 1/3 & 2 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Also find the median.

3. (*a*) Let X have the mgf

$$M(t) = e^{t^2/2}, -\infty < t < \infty$$

Find $E(X^{2k})$ and $E(X^{2k-1})$, for $k = 1, 2, 3, \ldots$

(*b*) Show by stating all the conditions that the Binomial distribution can be approximated to the Poisson distribution.

(*c*) Let X have the exponential pdf, $f(x) = \theta^{-1} \exp\{-x/\theta\}$, $0 < x < \infty$, zero elsewhere. Find the moment generating function of X, and hence, the mean, and the variance of X.

4. (*i*) Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = 15x_1^2x_2 \text{ if } 0 < x_1 < x_2 < 1$$
$$= 0 \text{ elsewhere}$$

Find the marginal pdf of X_1 and X_2 and compute $P(X_1 + X_2 \leq 1)$.

13/5/2019 (M)

This question paper contains 4+2 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **2710**

Unique Paper Code : **32357607** **IC**

Name of the Paper : **Probability Theory & Statistics**

Name of the Course : **B.Sc. (Hons.) Mathematics : DSE-3**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

In all there are *six* questions.

Question No. **1** is compulsory and it contains *seven* parts of **3** marks each, out of which any *five* parts are to be attempted.

In Question Nos. **2** to **6**, attempt any *two* parts from *three* parts.

Each part carries **6** marks.

Use of scientific calculator is allowed.

1. (*i*) If C_1, C_2 and C_3 are events in C, then prove that

$$p_1 \geq p_2 \geq p_3$$

where $p_1 = P(C_1) + P(C_2) + P(C_3)$,

$p_2 = P(C_1 \cap C_2) + P(C_2 \cap C_3) + P(C_1 \cap C_3)$, and

$p_3 = P(C_1 \cap C_2 \cap C_3)$.

(*ii*) Given the cumulative distribution function

$$F(x) = 0 \text{ if } x < -1$$

$$= (x+2)/4, \text{ if } -1 \le x < 1, \text{ and}$$

$$= 1 \text{ if } 1 \le x,$$

Compute :

(*i*) $P(-1/2 < X \le 1/2)$;

(*ii*) $P(X = 1)$.

(*iii*) Let pmf $p(x)$ be positive at $x = -1, 0, 1$ and zero elsewhere. If $p(0) = \frac{1}{4}$, find $E(X^2)$.

(*iv*) If the random variable X has a binomial distribution with the parameters n and θ, then compute the variance, σ^2, of X.

(*v*) Let $F(x, y)$ be the distribution function of X and Y. For all real constants $a < b$, $c < d$, show that

$$P(a < X \le b, c < Y \le d) = F(b, d) - F(b, c) - F(a, d) + F(a, c)$$

(*vi*) Let $f_{1/2}^{(x_1/x_2)} = \begin{cases} \dfrac{c_1 x_1}{x_2^2}, 0 < x_1 < x_2, \; 0 < x_2 < 1 \\ 0, \quad \text{elsewhere} \end{cases}$

be the conditional pdf of X_1 given $X_2 = x_2$.

Also let $f_2(x_2) = \begin{cases} c_2 x_2^4, \; 0 < x_2 < 1 \\ 0, \quad \text{elsewhere} \end{cases}$

be the marginal pdf of X_2.

Determine C_1 & C_2 and hence the joint pdf of X_1 and X_2.

(*vii*) Prove that $P_{i,j}^{n+m} = \Sigma_{k=0}^{\infty} P_{i,k}^{n} P_{k,j}^{m}$, for all n, m and all i, j.

2. (*a*) Let $\{C_n\}$ be a decreasing sequence of events, then show that

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P(\textstyle\bigcap_{n=1}^{\infty} C_n)$$

(*b*) In a lot of 50 light bulbs, there are 2 bad bulbs. An inspector examines five bulbs, which are selected at random and without replacement.

(*i*) Find the probability of at least one defective bulb among the five.

(*ii*) How many bulbs should be examined so that the probability of finding at least one bad bulb exceeds 1/2 ?

(*ii*) Suppose the joint mgf, $M(t_1, t_2)$, exists for the random variables X_1 and X_2. Then show that X_1 and X_2 are independent if and only if $M(t_1, t_2) = M(t_1, 0)\ M(0, t_2)$; that is, the joint mgf is identically equal to the product of the marginal mgfs.

(*iii*) Let X_1, X_2 be two random variables with joint $p(x_1, x_2) = \frac{1}{2^{x_1 + x_2}}$ for $1 \le x_i < \infty$, $i = 1, 2$, where x_1 and x_2 are integers, zero elsewhere. Determine the joint mgf of X_1, X_2 and show that X_1 and X_2 are independent random variables.

5. (*i*) Let X_1, X_2 be two random variables with joint pdf

$$f(x_1, x_2) = 4x_1x_2, \text{ if } 0 < x_1 < 1,\ 0 < x_2 < 1,$$
$$= 0 \text{ elsewhere}$$

(*a*) Is $E(X_1, X_2) = E(X_1)\ E(X_2)$?

(*b*) Find $E(3X_2 - 2X_1^2 + 6X_1X_2)$.

(*ii*) Suppose (X, Y) have a joint distribution with the variances of X and Y finite and positive. Denote the means and variances of X and Y by μ_1, μ_2 and σ_1^2, σ_2^2 respectively, and let ρ be the correlation coefficient between X and Y. If $E(Y \mid X = x)$ is linear in x, then

$$E(Y \mid X = x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x - \mu_1).$$

(*iii*) Let the random variables X and Y have the joint density function

$$f(x, y) = 1, \text{ if } -x < y < x,\ 0 < x < 1$$
$$= 0, \text{ elsewhere}$$

Show that, on the set of positive probability density, the graph of $E(Y \mid x)$ is a straight line, whereas that of $E(X \mid y)$ is not a straight line.

6. (*a*) (*i*) If X is a random variable with mean μ and variance σ^2, then prove that for any $k > 0$ $P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{\kappa^2}$.

(*ii*) Find the smallest value of k in above inequality for which the probability that a random variable will take a value between $(\mu - k\sigma)$ and $(\mu + k\sigma)$ is at least 0.99.

(*b*) State the Central limit theorem. Let X_i, $i = 1, 2, \ldots, 10$ be independent random variables, each having uniformly distributed over (0, 1). Estimate $P\{\Sigma_1^{10} X_i > 7\}$.

(*c*) An urn always contains 2 balls. Ball colors are red and blue. At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces. Define an appropriate Markov chain and if initially both balls are red, find the probability that the fifth ball selected is red.