(4)

(c) Explain why f(x) = sin(x) is a bounded function on R, whereas f(z) = sin(z) is not a bounded function on the complex plane C, although sin<sup>2</sup>(z) + cos<sup>2</sup>(z) = 1 for all z ∈ C.

2253

(a) State and prove Cauchy Integral formula.

4.

- (b) State Liouville's theorem and use it to prove the fundamental theorem of algebra.
- (c) Let C denote a contour of length L, and suppose that a function f(z) is piecewise continuous on C. Show that  $\left| \int_{C} f(z) dz \right| \le ML$ , where M is a non-negative constant such that  $|f(z)| \le M$  for all points z on C at which f(z) is defined. Hence, show that  $\left| \int_{C_{R}} \frac{2z^2 1}{z^4 + 5z^2 + 4} dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 1)(R^2 4)}$ , where  $C_{R}$  denote the upper half of the circle |Z| = R(R > 2), taken in the counterclockwise direction.

5. (a) Derive the expansions :

(i) 
$$\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} (0 < |z| < \infty);$$
  
(ii)  $z^3 \cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!}$   
 $\frac{1}{z^{2n-1}} (0 < |z| < \infty)$ 

Friday 10 5 2019 This question paper contains 4+2 printed pages] Roll No. S. No. of Question Paper : 2253 IC Unique Paper Code : 32351601 Name of the Paper **Complex Analysis B.Sc. (Hons.) Mathematics** Name of the Course VI Semester Maximum Marks: 75 **Duration: 3 Hours** (Write your Roll No. on the top immediately on receipt of this question paper.) Attempt all parts from Question No. 1. Each part carries 11/2 marks. Attempt any two parts from question Nos. 2 to 6 Each part carries six marks. State True or False. Justify your answer in brief : 1. A point  $z_0$  of a domain need not be an accumulation point (a)of that domain.  $\lim_{z \to 0} \frac{\overline{z}^2}{z} = 0$ (b)

P.T.O.

(c) The function  $f(z) = e^z$  is periodic with period  $2\pi$ .

2

)

2253

L

3.

(d)  $\log(-ei) = 1 - \frac{\pi}{2}i.$ 

162513

- (e) The function  $f(z) = |z|^2$  is analytic at z = 0.
- (f) Let C denote the boundary of the triangle with vertices at
  - the point 0, 3i, and -4, oriented in the counterclockwise

direction. Then 
$$\left| \int_{C} (e^{z} - \overline{z}) dz \right| \le 60.$$

- (g) If C is any simple closed contour, in either direction, then  $\int_{C} \exp(z^{3}) dz = 0.$
- (h) If C is the positively oriented unit circle |z| = 1, then

$$\int_{\mathcal{C}} \frac{\exp(2z)}{z^4} dz = \frac{8\pi i}{3}.$$

(i)  $\operatorname{Res}_{z=0} f(z) = -\frac{1}{3!}$ , where  $f(z) = z^2 \sin\left(\frac{1}{z}\right)$ ,  $0 < |z| < \infty$ .

(j) The function  $f(z) = \frac{1}{\sin(\frac{\pi}{z})}$  has no isolated singular

- 2. (a) Prove that a finite set of points cannot have any accumulation point.
  (b) Suppose that f(z) = u(x, y) + iv(x, y) (z = x + iy)
  - (b) Suppose that f(z) = u(x, y) + iv(x, y) (z = x + iy)and  $z_0 = x_0 + iy_0$ ,  $w_0 = u_0 + iv_0$ . Then  $\lim_{z \to z_0} f(z) = w_0$ if and only if  $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$  and  $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$ .
  - (c) Define neighbourhood of the point at infinity. Show that a set S is unbounded if and only if every neighbourhood of the point at infinity contains at least one point in S.
  - (a) Use Cauchy-Riemann equations to show that f'(z) does not exist at any point if f(z) = exp(z). State sufficient conditions for differentiability of a function f(z) at any point z<sub>0</sub> = x<sub>0</sub> + iy<sub>0</sub> ∈ C.
    - (b) Suppose that a function f(z) = u(x, y) + iv(x, y) and its conjugate f(z) = u(x, y) iv(x, y) are both analytic in a given domain D. Show that f(z) must be constant throughout D.

point.

(b) Give all the Laurent series expansions in powers of z for

the function  $f(z) = \frac{-1}{(z-1)(z-2)}$  and specify the domains in which those expansions are valid.

(c) If a power series  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$  converges when  $z = z_1(z_1 \neq z_0)$ , then show that it is absolutely convergent at each point z in the open disk  $|z - z_0| < R_1$  where  $R_1 = |z_1 - z_0|$ .

(a) Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(i) 
$$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$

(*ii*) 
$$f(z) = \frac{\sinh z}{z^4}$$

(*iii*) 
$$f(z) = \frac{\exp(2z)}{(z-1)^2}$$
.

(b)

6.

Use residues to evaluate  $\int_{0}^{2\pi} \frac{d\theta}{5+4\sin\theta}$ 

P.T.O.

(6)

If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then show

that 
$$\int_{C} f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$$
. Use it to show

that  $\int_{C} \frac{5z-2}{z(z-1)} dz = 10\pi i$ , where C is the circle |z| = 2,

6

described counterclockwise.

(c)