be an ordered basis of $\mathbb{R}^3$. Then, find $[L_A]_\beta$ and also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$, where L_A is a left-multiplication transformation. 5

This question paper contains 4 printed pages.

Your Roll No. ....

S. No. of Paper	: **6634**	**HC**
Unique paper code	: **32351403**	
Name of the paper	: **Ring Theory and Linear Algebra I**	
Name of course	: **B.Sc. (Hons.) Mathematics**	
Semester	: **IV**	
Duration	: **3 hours**	
Maximum marks	: **75**	

(Write your Roll No. on the top immediately on receipt of this question paper.)

*Attempt any **three** parts from each question.*

***All** questions are compulsory.*

1.(a) Prove that the intersection of any collection of subrings of a ring R is a subring of R. Is union of two subrings again a subring? Justify. 3+2=5

(b) Show that if an element a in $\mathbb{Z}_n$ is a unit, then a and n are relatively prime.

Show that for every prime p, $\mathbb{Z}_p$, the ring of integers modulo p, is a field if and only if p is a prime. 2+3=5

(c) Define characteristic of a ring. Show that the characteristic of an integral domain is either 0 or a prime. 1+4=5

(d) Show that the nilpotent elements of a commutative ring R form a subring of R. Let a be a nilpotent element of R, prove that $1 - a$ has a multiplicative inverse in R. 3+2=5

2. (a) Show that a maximal ideal is a prime ideal but not conversely. 3+2=5

(b) Let R be a ring with unity 1 and $\phi: R \to S$ be an onto ring homomorphism, then $\phi(1)$ is the unity of S, where $S \neq \{0\}$. What can you say if ϕ is not onto? Justify. 3+2=5

(c) Determine all ring homomorphisms from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$. 5

(d) Show that a field contains $\mathbb{Z}_p$ or $\mathbb{Q}$, where p is a prime. 5

3. (a) Prove that sum of two subspaces W_1 and W_2 of a vector space V is again a subspace of V. Further, show that $W_1 + W_2$ is the smallest subspace of V containing both W_1 and W_2. 5

(b) Show that for any subset S of a vector space V, span(S) is the smallest subspace of V containing S. 5

(c) Check whether $S = \{(0,1,-2),(1,-1,1),(1,2,1)\}$ forms a basis of $\mathbb{R}^3$ or not. 5

(d) Let V be a vector space over a field F and let S_1, S_2 be two subsets of V such that $S_1 \subseteq S_2 \subseteq V$. Then prove that:

(i) If S_1 is linearly dependent, then S_2 is also linearly dependent.

(ii) If S_2 is linearly independent, then so is S_1. 5

4. (a) Let V be a finite-dimensional vector space and if W be a subspace of V, then prove that any basis for W can be extended to a basis for V. 5

(b) Suppose that W be a subspace of a finite-dimensional vector space V, then W is finite-dimensional and $\dim(W) \leq \dim(V)$.

Further, if $\dim(W) = \dim(V)$, then $V = W$. 5

(c) Prove that the set of solutions to the system of linear equations :

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace. 5

(d) Extend the set $S = \{(1,0,0,0),(1,1,0,0),(0,0,1,1)\}$ to form a basis of $\mathbb{R}^4$. 5

5. (a) Let V and W be vector spaces and $T : V \to W$ be a linear transformation. Then prove that $N(T)$ and range $R(T)$ are subspaces of V and W, respectively. 5

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$T(x_1, x_2) = (2x_1 - x_2,\ 3x_1 + 4x_2,\ x_1).$$

Let β be the standard ordered basis for $\mathbb{R}^2$ and :

$$\gamma = \{(1,\ 1,\ 0),\ (0,\ 1,\ 1),\ (2,\ 2,\ 3)\}.$$

Compute $[T]_\beta^\gamma$. 5

(c) Let V be a vector space and let $T : V \to V$ be a linear transformation. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$, where $T_0(v) = 0$, for all $v \in V$. 5

(d) Let :

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

P. T. O.