6. (a) (i) Suppose that $f(x)=\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R>0$. Then show that $\int_0^x f(t)dt=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ for $|x|<R$ (5)

(ii) State when is a power series differentiable term by term. (1.5)

(b) (i) Apply Abel's Theorem to show :

$$\log_e 2 = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}.... \quad (3.5)$$

(ii) Given $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $|x|<1$. Evaluate:

$$\sum_{n=1}^{\infty}\frac{n}{5^n}, \quad \sum_{n=1}^{\infty}\frac{n^2(-1)^n}{3^n}. \quad (3)$$

(c) (i) App/ly Ratio Test for series to show that radius of convergence R of the power series $\sum a_n x^n$ is given by $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$, whenever the limit exists. (3.5)

(ii) Find radius of convergence for :

$$\sum_{n=2}^{\infty}(\ln(n))^{-1}x^n, \quad \sum_{n=0}^{\infty}\frac{(n!)^3 x^n}{(3n)!} \quad (3)$$

17/05/18 morning

This question paper contains 4 printed pages.

Your Roll No. ....

S. No. of Paper	: 6633	HC
Unique paper code	: 32351402	
Name of the paper	: **Reimann Integration and Series of Functions**	
Name of course	: **B.Sc. (Hons.) Mathematics**	
Semester	: **IV**	
Duration	: **3 hours**	
Maximum marks	: **75**	

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any two parts from each question.
All questions are compulsory.

Q.1 (a) Prove that a bounded function f on [a, b] is integrable if and only if for each $\varepsilon > 0$, $\exists$ a partition P of [a, b] such that $U(f, P) - L(f, P) < \varepsilon$. (6)

(b) Show that if f is integrable on [a, b], then $|f|$ is integrable on [a, b], and:

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Hence, show that :

$$\left|\int_{-2\pi}^{2\pi} x^2 Sin^8(e^x)dx\right| \le \frac{16\pi^3}{3}. \quad (6)$$

(c) Suppose f and g are continuous functions on [a, b], and $g(x) \ge 0$ $\forall x \in [a, b]$. Prove that, $\exists$ $x \in [a, b]$ such that:

$$\int_a^b f(t)g(t)\,dt = f(x)\int_a^b g(t)\,dt. \quad (6)$$

Q.2 (a) (i) The Dirichlet function $f: [0, 1] \to \mathbf{R}$ is defined by :

$$f(x)=\begin{cases} 1, & x\in[0,1]\cap\mathbb{Q} \\ 0 & x\in[0,1]\setminus\mathbb{Q}\end{cases}$$

Is f Riemann Integrable? Justify. (3)

(ii) Show that a decreasing function on [a, b] is integrable. (3)

(b) (i) State Fundamental Theorem of Calculus-I. (2)

(ii) Let f be an integrable function on [a, b]. Let P be a partition of [a,b] and P* be a refinement of P such that $P^* = P \cup \{u\}$. Show that $L(f, P) \le L(f, P^*)$. (4)

(c) (i) For a bounded function f on [a, b], define the Riemann Sum associated with a partition P. Hence, give Riemann's definition of integrability. (3)

(ii) Calculate $\lim_{h\to 0}\frac{1}{h}\int_3^{3+h} e^{t^2}\,dt$ (3)

3. (a) Show that:

$\beta(x,y)=\int_0^1 t^x(1-t)^y\,dt$ converges $\Leftrightarrow x>0, y>0$. (7)

(b) (i) Define Improper Integral of Type-II. When does it converge? When is it said to diverge ? (5)

(ii) Determine if $\int_{-\infty}^{\infty} e^x\,dx$ is a convergent integral. Justify. (2)

(c) (i) Determine if the following integrals are Improper Integrals. If so, what kind ? Justify.

$\int_0^{\pi/2} \frac{Sinx}{\sqrt[3]{x}}dx, \quad \int_0^1 x\ln(x)dx$ (4)

(ii) Prove that $\int_\pi^\infty \frac{Sinx}{x}dx$ converges absolutely. (3)

4. (a) State and prove Cauchy's criterion for uniform convergence for sequences of functions. (6)

(b) (i) Is the sequence $\{f_n\}$ where :

$$f_n = \frac{1}{n}\sin(nx+n).$$

uniformly convergent on **R** ? Justify. (2)

(ii) Suppose a sequence $\{f_n\}$ converges uniformly to f on a set A, and further suppose that each f_n is bounded on A. Show that the limit function f is bounded on A. (4)

(c) Show that if a>0, then the sequence $\{n^2x^2e^{-nx}\}$ converges uniformly on the interval $[a,\infty)$, but that it does not converge uniformly on the interval $[0,\infty)$. (6)

5. (a) If f_n is continuous on $D\subseteq \mathbf{R}$, for each $n\in\mathbb{N}$ and if Σf_n converges to f uniformly on D, then f is continuous on D. (6)

(b) Show that the series of functions $\Sigma\frac{x^n}{(1+x^n)}, x\ge 0$, converges uniformly on $[0, a]$ for $0<a<1$, but is not uniformly convergent on $[0, 1)$. (6)

(c) A sequence $\{f_n\}$ converges uniformly to f on A_0, if for each $\varepsilon>0$ there is a natural number $K(\varepsilon)$, depending only on ε, such that if $n\ge K(\varepsilon)$, then:

$$|f_n(x)-f(x)|<\varepsilon \quad \text{for all } x\in A_0.$$

Hence state a necessary and sufficient condition for a sequence $\{f_n\}$ to fail to converge uniformly on A_0 to f.

Apply this condition on sequence $f_n(x)=\frac{x}{n}$, for $x\in\mathbf{R}$ to examine for uniform convergence. (6)