6. (a) Using Trapezoidal rule approximate the value of the integral:

$$\int_0^2 \tan^{-1} x \, dx .$$

Further verify the theoretical error bound.

(b) Derive the closed Newton-Cotes rule $(n = 3)$ for the computation of the definite integral:

$$\int_a^b f(x)dx.$$

(c) Apply Euler's method to approximate the solution of the given initial value problem:

$$x' = \frac{1 + x^2}{t}, (1 \leq t \leq 4), x(1) = 0, N = 5.$$

Further it is given that the exact solution is:

$$x(t) = \tan(\ln(t)).$$

Compute the absolute error at each step. 12

10/12/18 (M)

This question paper contains 6 printed pages.

Your Roll No. ....

S. No. of Paper : **752** **I**

Unique Paper Code : **32357501**

Name of the Paper : **Numerical Methods**

Name of the Course : **B.Sc. (H) Mathematics : DSE-2**

Semester : **V**

Duration : **3 hours**

Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt all questions, selecting two parts from each question. Use of non-programmable scientific calculator is allowed.

1. (a) Given the following scheme for integration:

$$\int_a^b f(x)dx \approx \frac{h}{2} + [f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)],$$

write an algorithm to obtain the approximate value of the definite integral.

(b) Verify that the equation $x^5 - 2x - 1 = 0$ has a root in the interval (0, 1). Perform three iterations to approximate the zero of the equation by the Secant method using $p_0 = 0$ and $p_1 = 1$.

P. T. O.

(c) Let f be a continuous function on the closed interval $[a, b]$ and suppose that $f(a)f(b) < 0$.

Prove that the bisection method generates a sequence of approximations $\{p_n\}$ which converges to a root $p \in (a, b)$ with the property

$$|p_n - p| \leq \frac{b-a}{2^n}.$$

Hence, find the rate of the convergence of the method. 13

2. (a) Give the geometrical construction of the method of False Position to approximate the zero of a function. Further, write the algorithm for the computation of the root approximated by this method.

(b) Perform three iterations for finding the root of

$$f(x) = \frac{1}{x} - 37$$

by Newton's method starting with $p_0 = 0.01$. Further, compute the ratio

$$|p_3 - p|/|p_2 - p|^2$$

and show that this value approaches $|f''(p)/2f'(p)|$, with $p = 1/37$.

(ii) Define the backward difference operator and the central operator. Prove that:

$$\delta = \nabla (1 - \nabla)^{-1/2}.$$ 12

5. (a) Derive the formula:

$$f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2},$$

the second-order central difference approximation to the second order derivative of a function.

(b) Verify that:

$$f'(x) \approx \frac{f(x_0+h) - f(x_0-h)}{2h},$$

the difference approximation for the first order derivative provides the exact value of the derivative regardless of h, for the functions $f(x) = 1, f(x) = x$ and $f(x) = x^2$, but not for the function $f(x) = x^3$.

(c) Use the formula:

$$f'(x) \approx \frac{f(x_0) - f(x_0 - h)}{h}$$

to approximate the derivative of the function $f(x) = e^x$ at $x_0 = 0$, taking $h = 1, 0.1, 0.01$ and 0.001. What is the order of approximation? 12

$$\begin{bmatrix} 3 & 2 & -2 \\ -2 & -2 & 1 \\ 5 & -5 & 4 \end{bmatrix}.$$

(ii) Determine the spectral radius of the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 5 \end{bmatrix}.$$ 13

4. (a) Let $x_0, x_1, x_2, \dots, x_n$ be $n+1$ distinct points in $[a, b]$. If f is continuous on $[a, b]$ and has n continuous derivatives on (a, b), then prove that there exists $\xi \in (a, b)$ such that:

$$f\left[x_0, x_1, x_2, \dots, x_n\right] = \frac{f^n(\xi)}{n!}.$$

(b) Experimentally determined values for the partial pressure of water vapor, p_A, as a function of distance y, from the surface of a pan of water are given below. Estimate the partial pressure at distance 2.1 mm from the surface of the water.

y (mm)	0	1	2	3	4	5
p_A (atm)	0.10	0.065	0.042	0.029	0.022	0.020

(c) (i) Define an interpolating polynomial for a given set of data $(x_i, f(x_i))$, $i = 1, 2, \dots, n$. Construct the Lagrange polynomials passing through the points $(1, e)$, $(2, e^2)$ and $(3, e^3)$.

(c) Let g be a continuous function on the closed interval $[a, b]$ with $g: [a, b] \to [a, b]$. And suppose that g' is continuous on the open interval (a, b) with $|g'(x)| \le k < 1$ for all x belongs to (a, b). If $g'(p) \neq 0$, then prove that for any $p_0 \in [a, b]$, the sequence $p_n = g(p_{n-1})$ converges only linearly to the fixed point p. 13

3.(a) Using LU decomposition, solve the system of equations $Ax = b$, where:

$$A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} -3 \\ -12 \\ 6 \end{bmatrix}.$$

(b) Use the SOR method with $\omega = 0.9$ to solve the following system of equations:

$$2x_1 - x_2 = -1$$

$$-x_1 + 4x_2 + 2x_3 = 3$$

$$2x_2 + 6x_3 = 5$$

Use $x^{(0)} = \mathbf{0}$ and perform three iterations.

(c) (i) Compute the iteration matrix T_{gs} of the Gauss-Seidel method for obtaining the approximate solution of the system of equations $Ax = b$ where A is given as: