22/5/18

This question paper contains 5 printed pages.

Your Roll No. ....

S. No. of Paper	: **6631**	**HC**
Unique Paper Code	: **32351202**	
Name of the Paper	: **Differential Equations**	
Name of the Course	: **B.Sc. (Hons.) Mathematics – I**	
Semester	: **II**	
Duration	: **3 hours**	
Maximum Marks	: **75**	

(Write your Roll No. on the top immediately on receipt of this question paper.)

All the Sections are compulsory.
Use of non-programmable scientific calculator is allowed.

Section I

1. Attempt any *three* parts of the following: (5+5+5)

   a. Solve the differential equation:
   $$\left(2xe^{y}y^{4}+2xy^{3}+y\right)dx+\left(x^{2}e^{y}y^{4}-x^{2}y^{2}-3x\right)dy=0.$$

   b. Find the general solution of the differential equation:
   $$yy''+(y')^{2}=yy'.$$

   c. Solve the differential equation:
   $$\frac{dy}{dx}=\frac{x-y-1}{x+y+3}.$$

   d. Solve the initial value problem:
   $$\frac{dy}{dx}=2xy^{2}+3x^{2}y^{2},\ y(1)=-1.$$

2. Attempt any *two* parts of the following: (5+5)

   a. A water tank has the shape obtained by revolving the parabola $x^2 = by$ around the y axis. The water depth is 4 ft at 12 noon, when a circular plug in the bottom of the tank is removed. At 1 pm, the depth of the water is 1 ft. Find the

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depth $y(t)$ of water remaining after t hours. Also, find when the tank will be empty. If the initial radius of the top surface of the water is 2 ft, what is the radius of the circular hole in the bottom?

b. A certain piece of dubious information about phenyl ethyl amine in the drinking water began to spread one day in the city with a population of 100,000. Within a week 10,000 people heard this rumour. Assume that the rate of increase of the number who have heard the rumour is proportional to the number who have not heard it. How long will it be until half the population of the city has heard the rumour?

c. Consider a body that moves horizontally through a medium whose resistance is proportional to the square of the velocity υ, so that $d\upsilon/dt = -k\upsilon^2$. Show that :

$$\upsilon(t) = \frac{\upsilon_0}{1+\upsilon_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k}\ln(1+\upsilon_0 kt).$$

Section II

3. Attempt any *two* parts of the following: (8+8)

a. The following differential equation describes the level of pollution in the lake:

$$\frac{dC}{dt} = \frac{F}{V}(C_{in} - C)$$

where V is the volume, F is the flow (in and out), C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering the lake. Let $V = 28\times10^6\, m^3, F = 4\times10^6\, m^3/month$. If only fresh water enters the lake,

b. The predator-prey equations with additional deaths by DDT are:

$$\frac{dX}{dt} = \beta_1 X - c_1 XY - p_1 X, \frac{dY}{dt} = -\alpha_2 Y + c_2 XY - p_2 Y$$

where all parameters are positive constants.

i. Find all the equilibrium points.

ii. What effect does the DDT have on the non-zero equilibrium populations compared with the case when there is no DDT?

iii. Show that the predator fraction of the total average prey population is given by:

$$f = \frac{1}{1+\left(\frac{c_1(\alpha_2+p_2)}{c_2(\beta_1-p_1)}\right)}$$

What happens to this proportion f as the DDT kill rates, p_1 and p_2, increase?

c. The pair of differential equations

$$\frac{dP}{dt} = rP - \gamma PT, \frac{dT}{dt} = qP,$$

where r, γ and q are positive constants, is a model for a population of microorganisms P, which produces toxins T which kill the microorganisms.

i. Given that initially there are no toxins and P_0 microorganisms, obtain an expression relating the population density and the amount of toxins.

ii. Hence, give a sketch of a typical phase-plane trajectory.

iii. Using phase-plane trajectory, describe what happens to the microorganisms over time.

i. How long would it take for the lake with pollution concentration $10^7\ parts/m^3$ to drop below the safety threshold $(4\times 10^6\ parts/m^3)$?

ii. How long will it take to reduce the pollution level to 5% of its current level?

b. In view of the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is:

$$\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right) - h_0 X .$$

i. Show that the only non-zero equilibrium population is :

$$X_e = K\left(1-\frac{h}{r}\right).$$

ii. At what critical harvesting rate can extinction occur?

c. In a simple battle model, suppose that soldiers from the red army are visible to the blue army, but soldiers from the blue army are hidden. Thus, all the red army can do is fire randomly into an area and hope they hit something. The blue army uses aimed fire.

i. Write down appropriate word equations describing the rate of change of the number of soldiers in each of the armies.

ii. By making appropriate assumptions, obtain two coupled differential equations describing this system.

iii. Write down a formula for the probability of a single bullet fired from a single red soldier wounding a blue soldier in terms of the total area A and the area exposed by a single blue soldier A_b .

iv. Hence write the rate of wounding of both armies in terms of the probability and the firing rate.

Section III

4. Attempt any *three* parts of the following: (6+6+6)

a. Find the general solution of the differential equation:

$$x^3y''' + 6x^2y'' + 4xy' = 0$$

b. Using the method of undetermined coefficients, solve the differential equation :

$$y''' - 2y'' + y' = 1 + xe^x, y(0) = y'(0) = y''(0) = 1 .$$

c. Using the method of variation of parameters, solve the differential equation :

$$y'' + 3y' + 2y = 4e^x.$$

d. Show that $y_1 = 1$ *and* $y_2 = \sqrt{x}$ are solutions of :

$$yy'' + (y')^2 = 0,$$

but the sum $y = y_1 + y_2$ is not a solution. Explain why.

Section IV

5. Attempt any *two* parts of the following: (8+8)

a. Consider a disease where all those infected remain contagious for life. A model describing this is given by the differential equations

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI$$

where β is a positive constant.

i. Use the chain rule to find a relation between S and I.

ii. Obtain and sketch the phase-plane curves. Determine the direction of travel along the trajectories.

iii. Using this model, is it possible for all the susceptible to be infected?