[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 7114 **F-6** **Your Roll No................**

Unique Paper Code : 2351602 Paper Code : DC–I

Name of the Paper : ANALYSIS IV (Metric Spaces)

Name of the Course : **B.Sc. (Hons.) Mathematics (Erstwhile FYUP)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1. (a) Define a Metric Space. Let $X = \mathbb{R}^2$, for $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$, let $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Show that (X, d) is a metric space. (6½)

   (b) Let (X, d) be a metric space. Define $d^*: X \times X \to \mathbb{R}$ by

   $$d^*(x,y) = \min(1, d(x,y))$$

   for all $x, y \in X$. Show that (X, d^*) is a metric space. (6½)

(c) Define a complete metric space. Show that the metric space $(C[a,b],d)$ is complete, where $C[a,b]$ is the set of all continuous real-valued functions on $[a,b]$ and $d(f, g) = \sup\{|f(x) - g(x)|: a \leq x \leq b\}$, for all $f, g \in C[a,b]$. (6½)

2. (a) Let (X, d) be a metric space. Let A be a subset of X. Define Interior of A, Int (A). Show that Int (A) is an open subset of A that contains every open subset of A. (6½)

(b) Let (X,d) be a metric space and F be a subset of X. Define limit point of F. Show that the set of all limit points of F, namely F', is a closed subset of (X,d). (6½)

(c) Let (X,d) be a metric space. Show that if (X,d) is complete then for every nested sequence $\{F_n\}$ of non-empty closed subsets of X, where $d(F_n) \to 0$ as $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point. (6½)

3. (a) Let (X,d) be a metric space and F be a subset of X. Then show that the following statements are equivalent

(i) $x \in \overline{F}$

(ii) $S(x, \varepsilon) \cap F \neq \phi$ for every open ball $S(x, \varepsilon)$ centred at x

(iii) there exists an infinite sequence $\{x_n\}$ of points (not necessarily distinct) of F such that $x_n \to x$. (6½)

(b) Show that in any metric space (X,d) each open ball is an open set. Is the converse true ? Justify. (6½)

(c) Let Y be a subspace of a metric space (X,d). Show that every subset of Y that is open in Y is also open in X if and only if Y is open in X. (6½)

4. (a) If A is a subset of a metric space (X,d) then show that $d(A) = d(\bar{A})$, where d(A) denotes the diameter of the set A. (5)

(b) Let (X,d) and (Y,d^*) be metric spaces. Show that a mapping $f : X \to Y$ is continuous if and only if $f^{-1}(G)$ is open in X for every open subset G of Y. (5)

(c) Let (X,d) be a complete metric space and Y be a closed subset of X. Show that (Y,d_Y) is complete, where d_Y is the restriction of d to $Y \times Y$. (5)

5. (a) Let A and B be disjoint closed subsets of a metric space (X,d). Then show that there is a continuous real-valued function f on X such that $f(x) = 0$ for all $x \in A$, = $f(x) = 1$ for all $x \in B$ and $0 \leq f(x) \leq 1$ for all $x \in X$. (6½)

(b) State and prove contraction mapping principle. (6½)

(c) Define discrete two point space, (X_0,d_0). Let (X,d) be a metric space. Show that the following statements are equivalent :

(i) (X,d) is disconnected

(ii) there exists a continuous mapping of (X,d) onto (X_0,d_0). (6½)

6. (a) Let $(\mathbb{R},d)$ be the space of all real numbers with the usual metric. Show that every connected subset of $\mathbb{R}$ is an interval. (6½)

(b) Let (X,d) be a metric space and Y be a subset of X. Show that if Y is a compact subset of (X,d) then Y is bounded. Is the converse true ? Justify. (6½)

(c) Let f be a continuous function from a compact metric space (X,d) into an arbitrary metric space (Y,d^*). Show that f is uniformly continuous on X. (6½)