1/5/16

[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 7113 **F-6** **Your Roll No................**

Unique Paper Code : 2351601 (DC – I)

Name of the Paper : Algebra – V (Ring Theory and Linear Algebra II)

Name of the Course : **B.Sc. (Hons.) Mathematics (Erstwhile FYUP)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.

1. (a) If F is a field, then prove that F[x] is a principal ideal domain.

   (b) Let R be a commutative ring with unity. If I is a prime ideal of R, then prove that I[x] is a prime ideal of R[x].

   (c) Let F be a field and $p(x) \in F[x]$. Then $< p(x) >$ is a maximal ideal in F[x] if and only if p(x) is irreducible over F. (6.5,6.5,6.5)

2. (a) (i) Let F be a field and $f(x) \in F[x]$, where $\deg f(x) = 2$ or 3. Prove that f(x) is reducible over F if and only if f(x) has zero in F.

   (ii) Prove that, for every positive integer n, there are infinitely many polynomials of degree n in Z[x] that are irreducible over Q.

   (b) Define Euclidean domain. Prove that every Euclidean domain is a principal ideal domain.

(c) Prove that $Z\left[\sqrt{-5}\right]$ is not a unique factorization domain. (4+2,6,6)

3. (a) Let V be a finite dimensional vector space over a field F and V** be the double dual of V. Prove that V is isomorphic to V**.

(b) (i) Define the annihilator S^0 of subset S of a finite dimensional vector space V(F) and for subspaces W_1 and W_2 of V, show that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(ii) Let T be an invertible linear operator on a finite dimensional vector space V(F). Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(c) Let T be a linear operator on $P_2(R)$ defined by

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$

Test T for diagonalizability and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix. (6.5,3.5+3,6.5)

4. (a) Let T be a linear operator on R^4 defined by

$$T(a, b, c, d\} = (a + b + 2c - d,\ b + d,\ 2c - d,\ c + d)$$

and let $W = \{(t, s, 0, 0): t, s \in R\}$ be a subspace of R^4. Show that

(i) W is a T – invariant subspace of R^4.

(ii) the characteristic polynomial of T_W divides the characteristic polynomial of T.

(b) Let T be a linear operator on a finite dimensional vector space V over a field F and $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. If T is diagonalizable, then show that the minimal polynomial of T is of the form $p(t) = (t - \lambda_1)(t - \lambda_2) \ldots (t - \lambda_k)$.

(c) Let T be a linear operator on a finite dimensional vector space and p(t) be the minimal polynomial of T. If T is invertible and $p(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$, prove that $a_0 \neq 0$ and $T^{-1} = \frac{-1}{a_0}\left(T^{n-1} + a_{n-1}T^{n-2} + \cdots + a_2 T + a_1 I\right)$.

(2+4,6,6)

5. (a) Let $V = P_3(R)$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)\,dt$ and consider the subspace $P_2(R)$ of V with the standard basis $\{1, x, x^2\}$. Use Gram Scmidth process to obtain an orthonormal basis of $P_2(R)$. Also, compute the orthogonal projection of $f(x) = x^3$ on $P_2(R)$.

(b) Let V be a finite dimensional inner product space and W be a subspace of V. Prove that $V = W \oplus W^{\perp}$ where $W^{\perp}$ denotes the orthogonal complement of W.

(c) Prove that, if V is an inner product space, then

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \quad \forall x, y \in V$$

Verify the inequality for vectors $x = (1, 2i, 1 + i)$ and $y = (5 + i, 2, 3)$ in inner product space $\mathbb{C}^3$.

(6.5,6.5,6.5)

6. (a) Find the minimal solution of the following system of linear equations :

$$x + 2y + z = 4$$

$$x - y + 2y = -11$$

$$x + 5y = 19$$

(b) Let T be a linear operator on a finite dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular.

(c) (i) Let T be a linear operator on inner product space $\mathbb{C}^2$ defined by,

$$T(z_1, z_2) = (2z_1 + iz_2, (1 - i)z_1).$$

Evaluate T* at the vector $z = (3 - i, 1 + 2i)$.

(ii) Let T be a linear operator on a complex inner product space V with an adjoint T*. Prove that if T is self – adjoint, then $\langle Tx, x \rangle$ is real for all $x \in V$. (6,6,4+2)