(d) Let $f: R \to R$ be defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is differentiable at $x = 0$ and find $f'(0)$. (5)

5. (a) Let f be continuous on [a, b] and differentiate on (a, b). Prove that f is increasing on [a, b] if and only if $f'(x) \geq 0 \ \forall x \in [a, b]$. (5)

(b) State Darboux's Theorem. Suppose that if $f : [0,2] \to R$ is continuous on [0, 2] and differentiate on (0, 2), and that $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.

(i) Show that there exists $c_1 \in (0,1)$ such that $f'(c_1) = 1$

(ii) Show that there exists $c_2 \in (1,2)$ such that $f'(c_2) = 0$

(iii) Show that there exists $c \in (0,2)$ such that $f'(c) = 1/3$. (5)

(c) Find the Taylor series for cos x and indicate why it converges to cos x $\forall x \in R$. (5)

(d) Define a convex function on [a, b]. Check the convexity of the following functions on given intervals :

(i) $f(x) = x - \sin x$, $x \in [0, \pi]$.

(ii) $g(x) = x^3 + 2x$, $x \in [-1, 1]$. (5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 6623 **HC**

Unique Paper Code : 32351301

Name of the Paper : Theory of Real Functions

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **three** parts from each question.

1. (a) Use the ϵ-δ definition of the limit to find $\lim_{x \to 2} f(x)$ where $f(x) = \frac{1}{1-x}$. (5)

(b) State and prove Sequential Criterion for Limits. (5)

(c) State Squeeze Theorem. For $n \in N$, $n \geq 3$, derive the inequality, $-x^2 \leq x^n \leq x^2$ for $-1 < x < 1$. Hence prove that $\lim_{x \to 0} x^n = 0$ for $n \geq 3$, assuming that $\lim_{x \to 0} x^2 = 0$. (5)

(d) Let f, g be defined on $A \subseteq R$ to R, and let c be a cluster point of A. Suppose that f is bounded on a neighbourhood of c and that $\lim_{x \to c} g = 0$. Prove that $\lim_{x \to c} fg = 0$. (5)

2. (a) Let $c \in R$ and let f be defined for $x \in (c, \infty)$ and $f(x) > 0$ for all $x \in (c, \infty)$. Show that $\lim_{x \to c} f = \infty$ if and only if $\lim_{x \to c} \frac{1}{f} = 0$. (5)

(b) Prove that

(i) $\lim_{x \to 0} \frac{1}{\sqrt{|x|}} = +\infty, \ x \neq 0$

(ii) $\lim_{x \to 0-} e^{1/x} = 0, \ x \neq 0$. (5)

(c) Let A = R and let f be Dirichlet's function defined by

$$g(x) = \begin{cases} 1, & \text{for x rational} \\ -1, & \text{for x irrational} \end{cases}$$

Show that f is discontinuous at any point of R. (5)

(d) Let $f : R \to R$ be continuous at c and let $f(c) > 0$. Show that there exists a neighbourhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$ then $f(x) > 0$. (5)

3. (a) Determine the points of continuity of the function $f(x) = x - [\![x]\!]$, $x \in R$, where $[\![x]\!]$ denotes the greatest integer $n \in \mathbb{Z}$ such that $n \leq x$. (5)

(b) Let $A, B \subseteq R$, let $f: A \to R$ be continuous on A, and let $g : B \to R$ be continuous on B. If $f(A) \subseteq B$, show that the composite function $gof : A \to R$ is continuous on A. (5)

(c) Let f be a continuous real valued function defined on [a, b]. Show that f is a bounded function. (5)

(d) Prove that a polynomial of odd degree has at least one real root. (5)

4. (a) Define uniform continuity of a function on a set $A \subseteq R$. Show that every uniformly continuous function on A is continuous on A. Is the converse true? Justify your answer. (5)

(b) Show that the function $\sqrt{x}$ is uniformly continuous on $[0, \infty)$. (5)

(c) Let I, J be intervals in R, let $g : I \to R$ and $f : J \to R$ be functions such that $f(J) \subseteq I$ and let $c \in J$. If f is differentiate at c and if g is differentiate at f(c), show that the composite function gof is differentiate at c and $(gof)'(c) = g'(f(c)).f'(c)$. (5)