

- (i) Use the chain rule to find a relationship between R and B, given the initial numbers of soldiers for the two armies are r_0 and b_0 respectively.
- (ii) Draw a rough sketch of phase-plane trajectories.
- (iii) If both the army have equal attrition coefficients i.e. $a_1 = a_2$ and there are 10,000 soldiers in the red army and 8000 in blue army. Determine who wins if there is one battle between the two army.
- (c) Consider the Lotka - Volterra model describing the simple predator prey model :
- $$\frac{dx}{dt} = b_1X - c_1XY \quad \text{and} \quad \frac{dY}{dt} = c_2XY - a_2Y$$
- where b_1, c_1, c_2, a_2 are positive constants and X and Y denotes the prey and predator populations respectively at time t.
- (i) Find the equilibrium solutions of the above model.
- (ii) Find the directions of trajectories in the phase plane.

22/11/17 Morning
Monday

[This question paper contains 8 printed pages]

Your Roll No. :

Sl. No. of Q. Paper : 1824 **GC-4**

Unique Paper Code : 32351202

Name of the Course : **B.Sc.(Hons.)**
Mathematics-I

Name of the Paper : Differential Equations

Semester : II

Time : 3 Hours **Maximum Marks : 75**

Instructions for Candidates :

- (a) Write your Roll No. on the top immediately on receipt of this question paper.
- (b) Use of non-programmable scientific calculator is allowed.

SECTION - A

1. Attempt any **three** parts, each part is of **5** marks.

- (a) Solve the initial value problem :

$$x \frac{dy}{dx} + y = xy^{3/2}, y(1)=4.$$

- (b) Determine the most general function $M(x, y)$ such that the equation $M(x, y) dx + (x^2 y^3 + x^4 y) dy = 0$, is exact and hence solve it.

- (c) Solve the differential equation :

$$(x^2 - 3y^2) dx + 2xy dy = 0.$$

- (d) Check the exactness of the differential equation :

$$(3y + 4xy^2) dx + (2x + 3x^2y) dy = 0.$$

Hence solve it by finding the integrating factor of the form $x^p y^q$.

2. Attempt any **two** parts; each part is of **5** marks.

- (a) A certain moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half-life is about 1.28×10^9 years) that one of every nine potassium atoms disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium ?
- (b) A hemispherical bowl has top radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 inch is opened in the bottom of the tank. How long will it take for all the water to drain from the tank ?

SECTION - D

5. Attempt any **two** parts; each part is of **7.5** marks.

- (a) Consider a disease where all those who are infected remain contagious for life. Assume that there are no births and deaths :

(i) Write down suitable word equations for the rate of change of numbers of susceptibles and infectives. Hence develop a pair of differential equations.

(ii) Draw a sketch of typical phase-plane trajectories for this model. Determine the direction of travel along the trajectories.

- (b) A simple model for a battle between two army red and blue, where both the army used aimed fire, is given by the coupled differential equations -

$$\frac{dR}{dt} = -a_1 B, \quad \frac{dB}{dt} = -a_2 R$$

Where R and B are the number of soldiers in the red and blue army respectively and a_1 and a_2 are the positive constants.

- (b) Use the method of undetermined coefficients to solve the differential equation

$$y'' + y = \sin x.$$

- (c) A body with mass $m = \frac{1}{2}$ kg is attached to the end of the spring that is stretched 2 m (meters) by a force of 100 N (Newtons). It is set in motion with initial position $x_0 = 1$ m and initial velocity $v_0 = -5$ m/s. Find the position function of the body as well as the amplitude, frequency and period of the oscillation.

- (d) Show that the two solutions $y_1(x) = e^x \cos x$ and $y_2(x) = e^x \sin x$ of the differential equation $y'' - 2y' + 2y = 0$ are linearly independent on the open interval I . Then find a particular solution of the above differential equation with initial condition

$$y(0) = 1 \text{ and } y'(0) = 5.$$

- (e) Find the general solution of the Euler equation $x^2 y'' + 7xy' + 25y = 0$.

- (c) A motor boat starts from rest (initial velocity $v(0) = v_0 = 0$). Its motor provides a constant acceleration of 4 ft/s^2 , but water resistance causes a deceleration of $\frac{v^2}{400} \text{ ft/s}^2$. Find v when $t = 10$ s, and also find the limiting velocity as $t \rightarrow +\infty$ (that is, the maximum possible speed of the boat).

SECTION - B

3. Attempt any **two** parts; each part is of **7.5** marks.

- (a) Consider the American system of two lakes: Lake Erie feeding into Lake Ontario. Assuming that volume in each lake to remain constant and that Lake Erie is the only source of pollution for Lake Ontario.

- (i) Write down a differential equation describing the concentration of pollution in each of two lakes, using the variables V for volume, F for flow, $c(t)$ for concentration at time t and subscripts 1 for Lake Erie and 2 for Lake Ontario.

- (ii) Suppose that only unpolluted water flows into Lake Erie. How does this change the model proposed?
- (iii) Solve the system of equations to get an expression for the pollution concentration $c_1(t)$ and $c_2(t)$.
- (b) The following model describes the levels of a drug in a patient taking a course of cold pills:

$$\frac{dx}{dt} = I - k_1 x, \quad x(0) = 0$$

$$\frac{dy}{dt} = k_1 x - k_2 y, \quad y(0) = 0$$

Where k_1 and k_2 ($k_1 > 0$, $k_2 > 0$ and $k_1 \neq k_2$) describes rate at which the drug moves between the two sequential compartments (the GI-tract and the bloodstream) and I denotes the amount of drug released into the GI-tract in each step. At time t , x and y are the levels of the drug in the GI-tract and bloodstream respectively.

- (i) Find solution expressions for x and y which satisfies this pair of differential equations.
- (ii) Find the levels of the drug in the GI-tract and the bloodstream as $t \rightarrow \infty$.
- (c) In view of the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is
- $$\frac{dX}{dt} = rX \left(1 - \frac{X}{K} \right) - h_0 X.$$
- (i) Find the non-zero equilibrium population.
- (ii) At what critical harvesting rate can extinction occur?

SECTION - C

4. Attempt any **four** parts; each part is of **5** marks.
- (a) Use the method of variation of parameters to find a particular solution of the differential equation

$$y'' - 4y' + 4y = 2e^x.$$