(b) Prove that a bounded function f is integrable if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$. 6.5

(c) Show that the function f defined on $[-2, 2]$ as $f(x) = [x]$ is integrable and evaluate

$$\int_{-2}^{2} f(x)\,dx.$$ 6.5

This question paper contains 4 printed pages.

Your Roll No. ....

S. No. of Paper	: **3233**	**IC**
Unique Paper Code	: **62354443**	
Name of the Paper	: **Analysis**	
Name of the Course	: **B.A. (Prog.) Mathematics**	
Semester	: **IV**	
Duration	: **3 hours**	
Maximum Marks	: **75**	

(Write your Roll No. on the top immediately on receipt of this question paper.)

All six questions are compulsory.

Attempt any two parts from each question.

1. (a) Define the properties of real numbers as complete ordered field. 6

(b) Prove that if x and y are two positive real numbers, then there exists a positive integer n such that $nx > y$. 6

(c) Define an open set of real numbers. Prove that the intersection of a finite collection of open sets is open. What about the intersection of an infinite collection of open sets? Is it open or not? Justify. 6

P. T. O.

2. (a) Define limit point of a set $S \subseteq \mathbb{R}$. Show that zero is the only limit point of the set:

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$ 6

(b) Show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ -1, & \text{if } x \text{ is an irrational number} \end{cases}$$

is discontinuous at every real number x. 6

(c) Show that the function f defined by $f(x) = x^2$ is uniformly continuous on $[-2, 2]$. 6

3. (a) State Cauchy's convergence criterion for sequences and show that the sequence $< x_n >$, where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \forall n \in \mathbb{N}$$

is not convergent. 6.5

(b) Show that if $\lim_{n\to\infty} x_n = l$, where $x_n > 0 \ \forall n \in \mathbb{N}$ then:

$$\lim_{n\to\infty} (x_1 x_2 x_3 \cdots\cdots x_n)^{1/n} = l$$ 6.5

(c) Show that the sequence $< x_n >$ defined by :

$$x_1 = 1, \quad x_{n+1} = \sqrt{2 + x_n} \quad \forall n \in \mathbb{N}$$

is convergent. Find its limit. 6.5

4. (a) State and prove Cauchy's n^{th} root test for positive term series. 6.5

(b) Test the convergence of the following series:

(i) $\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$

(ii) $1 + \frac{x}{2} + \frac{x^2}{5} + \cdots + \frac{x^n}{n^2+1} \cdots\cdots$ for all positive values of x

(iii) $\frac{1}{3\cdot 7} + \frac{1}{4\cdot 9} + \frac{1}{5\cdot 11} + \frac{1}{6\cdot 13} \cdots\cdots$ 6.5

(c) Define absolute and conditional convergence of an alternating series. Prove that absolute convergence implies convergence but the converse is not true. 6.5

5. (a) Prove that every monotonic and bounded sequence converges. 6

(b) Find the limit superior and the limit inferior of each of the following sequences:

(i) $< 1 + (-1)^n >$

(ii) $< \frac{1}{n} >$

(iii) $< (-1)^n (1 + \frac{1}{n}) >$ 6

(c) Prove that every convergent sequence has a unique limit. 6

6. (a) Using integral test show that the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$ and divergent if $p \leq 1$. 6.5